

Compactified Jacobians of Integral Curves with Double Points

by

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Abstract

We study curves with double points and the compactified Jacobians and presentation schemes associated to them. If $\pi : C' \rightarrow C$ is the blow-up of a curve C with double points, then we show that the canonical morphism $\text{Pres}_\pi \rightarrow \bar{J}_{C'}$ is a \mathbb{P}^1 -bundle. We also present conditions under which the descent of line bundles through a birational morphism $f : X \rightarrow Y$ is controlled by the restriction of f over the subset of Y over which f is not an isomorphism, and we apply this result to the canonical birational morphism $\text{Pres}_\pi \rightarrow \bar{J}_C$. We then prove that the compactified Jacobian \bar{J}_C of a curve C with double points is canonically isomorphic to its Picard scheme, generalizing the corresponding classical result for smooth curves.

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INTRODUCTION

Compactified Jacobians of singular integral curves were initially introduced by Igusa [14] as limits of Jacobians of regular curves. Mayer and Mumford [19] gave the first intrinsic characterization of compactified Jacobians, in terms of torsion free, rank 1 sheaves, which were then constructed by D'Souza [5] using geometric invariant theory. In [2], Altman and Kleiman provided another treatment of compactified Jacobians, close in spirit to Grothendieck's construction of the relative Picard scheme in [9]. In this thesis, we obtain new results about compactified Jacobians of curves with certain singularities, namely, double points. In particular, we generalize the well-known classical result that the Jacobian of a regular curve is canonically isomorphic to its Picard scheme.

Let C be a non-singular integral curve of genus g and let J_C^n be its Jacobian, parametrizing invertible sheaves of Euler characteristic n on C . Then it is well-known that the pullback morphism

$$\Phi : \text{Pic}_{J_C^{-g}}^0 \longrightarrow J_C^{1-g}$$

induced by the canonical map $C \rightarrow J_C^{-g}$ sending P to $\mathcal{O}_C(-P)$ is an isomorphism. We refer to this property of the Jacobian of a non-singular curve as the *autoduality* of the Jacobian. If C is singular, of arithmetic genus p , then J_C^{-p} is no longer proper over the base, so that $\text{Pic}_{J_C^{-p}}$ is in general not representable. However, J_C^{-p} appears as an open subset of a natural compactification, the compactified Jacobian \bar{J}_C^{-p} . We can ask whether the similarly defined canonical map of Picard schemes

$$\Phi : \text{Pic}_{\bar{J}_C^{-p}}^0 \longrightarrow \bar{J}_C^{1-p}$$

is again an isomorphism. We answer this question affirmatively in Theorem 4.6 if all the singularities of C are double points.

An essential tool in the proof is the presentation scheme Pres_π associated to a partial normalization $\pi : C' \rightarrow C$ of C . Altman and Kleiman's paper [3] is a beautiful introduction to the presentation scheme, containing the seeds of many ideas in this thesis. The presentation scheme acts as a bridge between the compactified Jacobians of C' and C by mapping canonically to both. It is in general difficult, if not impossible, to give a satisfying geometric description of these maps. Nevertheless, concentrating on curves with double points, we can manage to understand the subtle interplay of the compactified Jacobians and the presentation scheme enough to prove autoduality. For instance, the canonical map $\kappa' : \text{Pres}_\pi \rightarrow \bar{J}_{C'}$ is shown in Proposition 3.14 to make Pres_π a \mathbb{P}^1 -bundle over $\bar{J}_{C'}$. This allows us to relate line bundles on $\bar{J}_{C'}$ to line bundles on Pres_π .

On the other hand, the geometry of the birational morphism $\kappa : \text{Pres}_\pi \rightarrow \bar{J}_C$ is substantially more complex and our understanding of it correspondingly partial. In Proposition 3.9

we give a description of κ , but only in the smooth topology and over an open subset \mathcal{S}_π^1 of \bar{J}_C . Working in the smooth topology does not create any difficulties since the properties of κ we are concerned with are stable under faithfully flat base change. But *a priori*, restricting to the open subset \mathcal{S}_π^1 of \bar{J}_C might seem rather severe. Remarkably, this is not the case: \mathcal{S}_π^1 has just the right size. Indeed, its complement has codimension at least 2 inside \bar{J}_C , according to (3.6). It is thus large enough to retain enough information about the whole of \bar{J}_C , as expressed in Lemma 1.17. At the same time, it is small enough to allow us to compare line bundles on \mathcal{S}_π^1 to line bundles on the inverse image \mathcal{T}_π^1 of \mathcal{S}_π^1 in Pres_π via a comparatively well-understood faithfully flat morphism $\bar{\kappa}$.

We achieve this comparison using the more general Theorem 1.11. Given a finite surjective birational morphism $f : X \rightarrow Y$ of integral schemes, put $\mathcal{B} = f_*\mathcal{O}_X/\mathcal{O}_Y$, and let $\Sigma = \text{Supp } \mathcal{B}$ so that f is an isomorphism exactly over the open complement of Σ ; we can call Σ the *sticking locus* of f . We restrict f to $\bar{f} : f^{-1}(\Sigma) = \Sigma' \rightarrow \Sigma$ and attach to this restricted map \bar{f} two canonical functors $\underline{G}_{\bar{f}}$ and $\underline{H}_{\bar{f}}$. If \mathcal{B} is locally free on its support Σ , then \bar{f} is faithfully flat and finite and the pullback f^* fits in an exact sequence

$$0 \longrightarrow \underline{G}_{\bar{f}} \longrightarrow \underline{\text{Pic}}_Y \xrightarrow{f^*} \underline{\text{Pic}}_X \longrightarrow \underline{H}_{\bar{f}}.$$

To prove autoduality, we first construct a one-sided inverse Ψ to Φ and then proceed by induction on the genus deficiency δ of C , which is equal to the difference between the arithmetic and the geometric genera of C . If $\delta = 0$, then C is smooth and autoduality is classical. Otherwise we relate the compactified Jacobian of C to that of some partial normalization C' , for which autoduality holds by the induction hypothesis, through the presentation scheme and apply our knowledge of the maps κ and κ' to calculate that Ψ must indeed be an isomorphism.

We can contrast this autoduality we prove to the stronger, albeit conjectural, assertion that the *compactified* Picard schemes $\text{Pic}_{\bar{J}_C}^{0=p}$ and \bar{J}_C^{1-p} are isomorphic. However, it is not yet known whether Φ even extends to a morphism $\text{Pic}_{\bar{J}_C}^{0=p} \rightarrow \bar{J}_C^{1-p}$.

It is of course natural to ask whether autoduality holds for curves with singularities other than double points. There is on the one hand no example yet to the contrary. However, a surprisingly large number of properties of double points used in the proofs that follow actually *characterize* double points. For instance, one benefits significantly from the seemingly casual fact that any partial normalization of a curve with double points is locally planar. However, an ordinary triple point already has a partial normalization which is not even Gorenstein! See Proposition 3.3. Hence, unless a coarse induction can be used which effectively skips some partial normalizations, any proof of autoduality for curves with singularities other than double points (if such is true) will probably have to take into account non-locally planar curves, the compactified Jacobians of which are of a substantially more

complex nature than the ones we deal with in this thesis due to the fact that they are not irreducible.

We now present a summary of the contents of this thesis.

In Section 1, we study the descent of line bundles through a non-flat birational morphism and show how, under suitable conditions, this descent can be understood in terms of a simpler faithfully flat morphism.

In Section 2, we mostly introduce the notation used in the sequel and recall well-known results about compactified Jacobians and the presentation scheme, while also presenting some new general results.

In Section 3, we explore in some detail double points of curves and the compactified Jacobians and presentation schemes of curves with double points.

In Section 4, we use the previous results to prove the autoduality of the compactified Jacobian of a curve with double points.

1. DESCENT FOR BIRATIONAL MORPHISMS

(1.1) We work over a connected Noetherian base scheme S . An S -variety is a flat morphism $\sigma_X : X \rightarrow S$ with integral geometric fibers. We do not assume that X is projective or even proper over S .

For any S -variety X , we let $\underline{\text{Pic}}_{X/S}$ denote the relative Picard functor of X over S , as defined in [9]. If X is projective over S , this functor is representable by a scheme $\text{Pic}_{X/S}$, called the relative Picard scheme of X over S ; see [9, Théorème 3.1]. As usual, $\text{Pic}_{X/S}^0$ denotes the subscheme of $\text{Pic}_{X/S}$ defined by the condition that the fiber of $\text{Pic}_{X/S}^0$ over s be equal to the connected component of the fiber of $\text{Pic}_{X/S}$ containing \mathcal{O}_{X_s} , for any $s \in S$.

Let X and Y be S -varieties. A morphism $f : X \rightarrow Y$ is called *relatively birational* if the induced morphism $f(s) : X(s) \rightarrow Y(s)$ on the fiber over any $s \in S$ is birational. Clearly, any base extension of a relatively birational morphism is relatively birational. We will exclusively be concerned with *finite surjective* relatively birational morphisms. Note that a relatively birational morphism of *projective* S -varieties is automatically surjective and is finite as soon as it is quasi-finite. For any finite surjective relatively birational morphism $f : X \rightarrow Y$ of S -varieties, set $\mathcal{B}_f = f_*\mathcal{O}_X/\mathcal{O}_Y$. Since f is affine and X is flat over S , we see that $f_*\mathcal{O}_X$ is flat over S . Furthermore, the comorphism $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is injective on the fibers since f is relatively birational and surjective. Hence by [11, 8.11.1] the cokernel \mathcal{B}_f is flat over S , its formation commutes with base change, and the comorphism is injective. We also set $\Sigma_f = \text{Supp}(\mathcal{B}_f)$ and $\Sigma'_f = f^{-1}(\Sigma_f)$ and let

$$\bar{f} = f|_{\Sigma'_f} : \Sigma'_f \rightarrow \Sigma_f$$

be the restricted morphism. When there is no possible confusion, we omit the subscript f .

Finally, we let $R = X \times_Y X$ and $g : R \rightarrow Y$ be the composite morphism, $p_i : R \rightarrow X$ be the projections, and $\Delta : X \rightarrow R$ the diagonal. Let $i : \Sigma \hookrightarrow Y$ and $i' : \Sigma' \hookrightarrow X$ be the inclusions, let $\Sigma'' = \Sigma' \times_\Sigma \Sigma'$, with $\delta : \Sigma' \hookrightarrow \Sigma''$ the diagonal and $j : \Sigma'' \hookrightarrow R$ the canonical inclusion. Let $\pi_i : \Sigma'' \rightarrow \Sigma'$ be the projections.

Lemma 1.2. *Let $f : X \rightarrow Y$ be a finite surjective relatively birational morphism of S -varieties. Suppose that the sheaf \mathcal{B} on Y supported on Σ is locally free of rank r on Σ . Then*

1. Σ is flat over S and Σ' is faithfully flat and finite of degree $r + 1$ over Σ .

2. The sequence

$$(1.3) \quad 0 \longrightarrow \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X \xrightarrow{p_1 - p_2} g_* \mathcal{O}_R$$

is exact and $\text{Coker}(p_1 - p_2) = f_* \mathcal{O}_X \oplus \mathcal{F}$, where \mathcal{F} is locally free of rank r on Σ .

3. The sequence

$$(1.4) \quad 0 \longrightarrow \mathcal{O}_R \xrightarrow{\alpha} \Delta_* \mathcal{O}_X \times j_* \mathcal{O}_{\Sigma''} \xrightarrow{\beta} (j \circ \delta)_* \mathcal{O}_{\Sigma'} \longrightarrow 0$$

of \mathcal{O}_R -modules is exact, where α is induced by $(\Delta, j) : X \amalg \Sigma'' \rightarrow R$ and β is induced by $\delta : \Sigma' \rightarrow \Sigma''$ and $i' : \Sigma' \rightarrow X$.

PROOF. We already know that \mathcal{B} is flat over S and that the comorphism $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is injective. By hypothesis, \mathcal{B} is locally isomorphic to $\mathcal{O}_\Sigma^{\oplus r}$ on Y , so higher Tor's of \mathcal{O}_Σ must vanish since they commute with direct sums in the first variable and \mathcal{B} is flat. Hence Σ is flat over S . We have the following sequence of \mathcal{O}_Σ -modules:

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \bar{f}_* \mathcal{O}_{\Sigma'} \longrightarrow \mathcal{B} \longrightarrow 0.$$

Since \mathcal{B} is locally free of rank r , we see that $\bar{f}_* \mathcal{O}_{\Sigma'}$ is locally free of rank $r + 1$, so that \bar{f} is flat.

Proving the exactness of (1.4) is equivalent to proving the exactness of

$$(1.5) \quad 0 \longrightarrow g_* \mathcal{O}_R \xrightarrow{\alpha} f_* \mathcal{O}_X \times (g \circ j)_* \mathcal{O}_{\Sigma''} \xrightarrow{\beta} (f \circ (i'))_* \mathcal{O}_{\Sigma'} \longrightarrow 0$$

on Y since g is affine and surjective. The exactness of the sequences (1.3) and (1.5) is clear off Σ , since $X - \Sigma' \simeq Y - \Sigma$. Let $P \in \Sigma \subseteq Y$, let $R = \mathcal{O}_{Y,P}$ and $R' = (f_* \mathcal{O}_X)_P$. Let $I = \text{Ann}_R(R'/R)$ be the conductor, so that by hypothesis, we have $R'/R \simeq (R/I)^{\oplus r}$ as

R -modules. We have the sequence of R -modules

$$(1.6) \quad 0 \longrightarrow R \longrightarrow R' \longrightarrow R'/R \longrightarrow 0.$$

Tensoring it with R' , we get the sequence

$$R' \longrightarrow R' \otimes_R R' \longrightarrow R' \otimes_R R'/R \longrightarrow 0,$$

which is split by the surjective diagonal $R' \otimes_R R' \rightarrow R'$, so that

$$(1.7) \quad R' \otimes_R R' \simeq R' \oplus (R' \otimes_R R'/R).$$

The map $p_1 - p_2 : R' \rightarrow R' \otimes_R R'$ factors through the second summand by sending $b \in R'$ to $1 \otimes \bar{b}$, where \bar{b} is the canonical image of b in R'/R . Let $b \in R'$ and let \bar{u} in $(R/I)^{\oplus r}$ correspond to \bar{b} in R'/R under the isomorphism $R'/R \simeq (R/I)^{\oplus r}$. Consider the composition

$$\begin{array}{ccccccc} R' & \longrightarrow & R' \otimes_R R'/R & \xrightarrow{\sim} & R' \otimes_R (R/I)^{\oplus r} & \xrightarrow{\sim} & (R'/I)^{\oplus r} \\ & & b & & 1 \otimes \bar{b} & & 1 \otimes \bar{u} & & \bar{u} \end{array}$$

We have $1 \otimes \bar{b} = 0$ if and only if $\bar{u} = 0$, which will happen if and only if $\bar{b} = 0$. This is equivalent to $b \in R$, which proves exactness of (1.3). Furthermore, the cokernel of the composition is $(R/I)^{\oplus r}$, so that \mathcal{F} is locally free of rank r on Σ .

We now have to show that the sequence

$$0 \longrightarrow R' \otimes_R R' \xrightarrow{\alpha} R' \times (R'/I \otimes_{R/I} R'/I) \xrightarrow{\beta} R'/I \longrightarrow 0$$

of R -modules is exact, where $\alpha(\sum x_i \otimes y_i) = (\sum x_i y_i, \sum \bar{x}_i \otimes \bar{y}_i)$ and $\beta(w, \sum \bar{x}_i \otimes \bar{y}_i) = w - \sum \bar{x}_i \bar{y}_i$. Clearly $\beta(0, \bar{x} \otimes 1) = \bar{x}$, so that β is surjective. It is easy to see that $\beta \circ \alpha = 0$. Conversely, if $(w, \sum \bar{x}_i \otimes \bar{y}_i)$ is such that $\bar{w} = \sum \bar{x}_i \bar{y}_i$, let $u = w - \sum x_i y_i$, so that $u \in I$. Then $\alpha(\sum x_i \otimes y_i + u \otimes 1) = (w, \sum \bar{x}_i \otimes \bar{y}_i)$, so that $\text{Im } \alpha = \text{Ker } \beta$. It remains to show that α is injective. Let $\{\bar{e}_0 = 1, \bar{e}_1, \dots, \bar{e}_r\}$ be a basis for R'/I over R/I . By Nakayama's lemma, R' is generated over R by the e_i 's. Any $w \in R' \otimes_R R'$ can be written as $w = \sum a_{ij} e_i \otimes e_j$ with a_{ij} in R . Suppose $\alpha(w) = 0$. Then we must have $\sum \bar{a}_{ij} \bar{e}_i \otimes \bar{e}_j = 0$, and since $\{\bar{e}_i \otimes \bar{e}_j\}_{ij}$ forms a basis of $R'/I \otimes_{R/I} R'/I$ over R/I , we must have that $\bar{a}_{ij} = 0$, that is, $a_{ij} \in I$. Then, by definition of I , we have that $a_{ij} e_j \in R$, so we can commute those quantities across the tensor sign, which allows us to write $w = \sum a_{ij} (e_i e_j \otimes 1)$. But by the definition of α , and since $\alpha(w) = 0$, we must also have $\sum a_{ij} e_i e_j = 0$, which proves the lemma. \square

(1.8) Theorem 1.11 below establishes that, under the conditions of Lemma 1.2 above, the descent of line bundles through the (non-flat) relatively birational morphism $f : X \rightarrow Y$ is almost entirely controlled by the (flat) restricted morphism $\bar{f} : \Sigma' \rightarrow \Sigma$. We first need to introduce some notation. For any line bundle \mathcal{L} on X , we have the sequence of inclusions

$$\text{EDes}_f(\mathcal{L}) \subseteq \text{Des}_f(\mathcal{L}) \subseteq \text{Cov}_f(\mathcal{L}) \subseteq \text{Hom}_{\mathcal{O}_R}(p_1^*\mathcal{L}, p_2^*\mathcal{L}),$$

where

- $\text{Cov}_f(\mathcal{L})$ denotes the set of *covering data* of \mathcal{L} for f , that is, the set of isomorphisms $\phi : p_1^*\mathcal{L} \xrightarrow{\sim} p_2^*\mathcal{L}$.
- $\text{Des}_f(\mathcal{L})$ denotes the set of *descent data* of \mathcal{L} for f , that is, those covering data ϕ that satisfy the cocycle condition

$$p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi$$

on the triple product $X \times_Y X \times_Y X$.

- $\text{EDes}_f(\mathcal{L})$ denotes the set of *effective* descent data of \mathcal{L} for f , where a descent datum $\phi \in \text{Des}_f(\mathcal{L})$ is said to be effective if there exists an invertible sheaf \mathcal{N} on Y and an isomorphism $\theta : f^*\mathcal{N} \xrightarrow{\sim} \mathcal{L}$ such that the following diagram commutes:

$$(1.9) \quad \begin{array}{ccccc} p_1^*f^*\mathcal{N} & \longrightarrow & g^*\mathcal{N} & \longrightarrow & p_2^*f^*\mathcal{N} \\ p_1^*\theta \downarrow & & & & \downarrow p_2^*\theta \\ p_1^*\mathcal{L} & \xrightarrow{\phi} & & & p_2^*\mathcal{L}. \end{array}$$

Recall that $f : X \rightarrow Y$ is a *descent morphism* for invertible sheaves if for any pair \mathcal{L}, \mathcal{M} of invertible sheaves on Y , the sequence

$$(1.10) \quad 0 \longrightarrow \text{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{M}) \longrightarrow \text{Hom}_{\mathcal{O}_X}(f^*\mathcal{L}, f^*\mathcal{M}) \longrightarrow \text{Hom}_{\mathcal{O}_R}(g^*\mathcal{L}, g^*\mathcal{M})$$

is exact. It is a *strict* descent morphism if in addition any descent datum is effective, that is, if for any line bundle \mathcal{L} on X , we have $\text{EDes}_f(\mathcal{L}) = \text{Des}_f(\mathcal{L})$. For example, faithfully flat maps are strict descent morphisms; see [8].

We can turn these notions into functors by letting the group functors on S -schemes \underline{G}_f and \underline{EG}_f be defined by

$$\underline{G}_f(T) = \text{Des}_{f_T}(\mathcal{O}_{X_T})$$

and

$$\underline{EG}_f(T) = \text{EDes}_{f_T}(\mathcal{O}_{X_T}).$$

Note that if f is a strict descent morphism, then $\underline{EG}_f = \underline{G}_f$, since the notion is stable under base change. Let $\underline{K}_f = \text{Ker } f^* : \underline{\text{Pic}}_{Y/S} \rightarrow \underline{\text{Pic}}_{X/S}$. There is a close relationship between \underline{K}_f and \underline{EG}_f given a canonical map

$$\underline{EG}_f \rightarrow \underline{K}_f$$

associating to an effective descent datum $\phi \in \text{EDes}_{f_T}(\mathcal{O}_{X_T})$ the invertible sheaf \mathcal{N} on Y_T such that $f_T^* \mathcal{N} \xrightarrow{\sim} \mathcal{O}_{X_T}$. The latter morphism is well defined since \mathcal{N} is the kernel of the canonical morphism

$$p_1 - \phi^{-1} p_2 : f_* \mathcal{L} \rightarrow g_*(p_1^* \mathcal{L}).$$

It is furthermore surjective since choosing \mathcal{L} on Y_T and an isomorphism $\theta : f^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{X_T}$ defines a descent datum ϕ which is by definition effective, and which maps to \mathcal{L} by the above map.

We can also introduce the above notations for the restricted morphism $\bar{f} : \Sigma' \rightarrow \Sigma$ and a line bundle $\bar{\mathcal{L}}$ on Σ' . Note, however, that Σ and Σ' do not, in general, have integral fibers over S , hence are not S -varieties. This fact is reflected in the relationship between \underline{K}_f and \underline{EG}_f as follows. Suppose, as is often the case, that the relation $f_* \mathcal{O}_X = \mathcal{O}_S$ holds universally (for example, if X is proper over S). Then we claim that $\underline{EG}_f \xrightarrow{\sim} \underline{K}_f$. To prove this, let $\phi \in \underline{EG}_f(T)$ be such that it gives rise to the trivial line bundle \mathcal{O}_{Y_T} . Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{R_T} & \xrightarrow{\phi} & \mathcal{O}_{R_T} \\ p_1^* \theta \downarrow & & \downarrow p_2^* \theta \\ \mathcal{O}_{R_T} & \xrightarrow{\text{id}} & \mathcal{O}_{R_T}, \end{array}$$

for some $\theta \in H^0(X_T, \mathcal{O}_{X_T}^*)$. But since $f_* \mathcal{O}_X = \mathcal{O}_S$ holds universally, θ must be a unit on T . In particular, $p_1^* \theta = p_2^* \theta$, which shows that $\phi = \text{id}$. On the other hand, the morphism $\bar{f} : \Sigma' \rightarrow \Sigma$ behaves very differently. For example, it is in general faithfully flat and has a section, so that $\underline{K}_{\bar{f}} = \{\mathcal{O}_{\Sigma}\}$; but $\underline{EG}_{\bar{f}}$ can be large.

Define finally the group functor $\underline{H}_{\bar{f}}$ to be the cokernel of \bar{f}^* , that is,

$$\underline{H}_{\bar{f}}(T) = \underline{\text{Pic}}_{\Sigma'/S}(T) / \bar{f}^* \underline{\text{Pic}}_{\Sigma/S}(T).$$

Theorem 1.11. *Let $f : X \rightarrow Y$ be a finite surjective relatively birational morphism of S -varieties. Suppose that the sheaf \mathcal{B} on Y supported on Σ is locally free on Σ . Then the morphism f is a strict descent morphism and there is a canonical isomorphism $\underline{G}_f \xrightarrow{\sim} \underline{G}_{\bar{f}}$.*

Moreover, suppose that $(\sigma_Y)_* \mathcal{O}_Y = \mathcal{O}_S$ holds universally. Then the following sequence of functors is exact in the fppf topology:

$$(1.12) \quad 0 \longrightarrow \underline{G}_{\bar{f}} \longrightarrow \underline{\mathrm{Pic}}_{Y/S} \longrightarrow \underline{\mathrm{Pic}}_{X/S} \longrightarrow \underline{H}_{\bar{f}}.$$

If X has a section over S , then the sequence is exact in the Zariski topology.

PROOF. Let \mathcal{L} and \mathcal{M} be line bundles on Y and let $\mathcal{N} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{M})$, which is flat on Y . Tensoring sequence (1.3) with \mathcal{N} and using the projection formula, we obtain the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow f_* f^* \mathcal{N} \longrightarrow g_* g^* \mathcal{N}.$$

Taking global sections, we obtain the sequence (1.10) as desired, so f is a descent morphism. Proving it is strict is rather more delicate.

We first prove that for any line bundle \mathcal{L} on X , the pullback map

$$(i')^* : \mathrm{Cov}_f(\mathcal{L}) \longrightarrow \mathrm{Cov}_{\bar{f}}(\bar{\mathcal{L}})$$

restricted to descent data induces an isomorphism

$$(1.13) \quad (i')^* : \mathrm{Des}_f(\mathcal{L}) \xrightarrow{\sim} \mathrm{Des}_{\bar{f}}(\bar{\mathcal{L}}),$$

where $\bar{\mathcal{L}} = (i')^* \mathcal{L}$. Choose $\bar{\phi} \in \mathrm{Cov}_{\bar{f}}(\bar{\mathcal{L}})$. Tensoring sequence (1.4) successively with $p_1^* \mathcal{L}$ and $p_2^* \mathcal{L}$, we can form the diagram

$$(1.14) \quad \begin{array}{ccccccc} 0 & \longrightarrow & p_1^* \mathcal{L} & \longrightarrow & \Delta_* \mathcal{L} \oplus j_* \pi_1^* \bar{\mathcal{L}} & \longrightarrow & (j \circ \delta)_* \bar{\mathcal{L}} \longrightarrow 0 \\ & & & & \downarrow (\mathrm{id}, j_* \bar{\phi}) & & \downarrow \mathrm{id} \\ 0 & \longrightarrow & p_2^* \mathcal{L} & \longrightarrow & \Delta_* \mathcal{L} \oplus j_* \pi_2^* \bar{\mathcal{L}} & \longrightarrow & (j \circ \delta)_* \bar{\mathcal{L}} \longrightarrow 0, \end{array}$$

so there is an induced isomorphism $\phi : p_1^* \mathcal{L} \xrightarrow{\sim} p_2^* \mathcal{L}$. We thus get a map $\nu : \mathrm{Cov}_{\bar{f}}(\bar{\mathcal{L}}) \rightarrow \mathrm{Cov}_f(\mathcal{L})$, which sends $\bar{\phi}$ to ϕ . We claim that this map inverts $(i')^*$ on descent data. Pulling back diagram (1.14) to Σ'' by j , we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1^* \bar{\mathcal{L}} & \longrightarrow & \delta_* \bar{\mathcal{L}} \oplus \pi_1^* \bar{\mathcal{L}} & \longrightarrow & \delta_* \bar{\mathcal{L}} \longrightarrow 0 \\ & & & & \downarrow (\mathrm{id}, \bar{\phi}) & & \downarrow \mathrm{id} \\ 0 & \longrightarrow & \pi_2^* \bar{\mathcal{L}} & \longrightarrow & \delta_* \bar{\mathcal{L}} \oplus \pi_2^* \bar{\mathcal{L}} & \longrightarrow & \delta_* \bar{\mathcal{L}} \longrightarrow 0. \end{array}$$

It is easy to see that the maps $\pi_i^* \bar{\mathcal{L}} \rightarrow \pi_i^* \bar{\mathcal{L}}$ appearing in the horizontal sequences for $i = 1, 2$ are both the identity, so we recover $\bar{\phi} : \pi_1^* \bar{\mathcal{L}} \rightarrow \pi_2^* \bar{\mathcal{L}}$, which proves that $(i')^* \circ \nu = \mathrm{id}$. On

the other hand, if we start with $\phi \in \text{Cov}_f(\mathcal{L})$ such that $\Delta^*\phi = \text{id}$ and let $\bar{\phi} = (i')^*\phi$, then both ϕ and $\nu(\bar{\phi})$ fit in diagram (1.14), so they must be equal. Hence ν exactly inverts $(i')^*$ on the subset of all covering data which are such that they pull back to the identity by the diagonal Δ . *A fortiori*, our claim is proved, since $(i')^*$ sends descent data to descent data. Base changing through any morphism $T \rightarrow S$, we see that we have an isomorphism $\underline{G}_f \rightarrow \underline{G}_{\bar{f}}$.

Let \mathcal{L} be a line bundle on X and $\phi \in \text{Des}_f(\mathcal{L})$. Let

$$p_\phi = p_1 - \phi^{-1} \circ p_2 : f_*\mathcal{L} \rightarrow g_*(p_1^*\mathcal{L}).$$

Let $\mathcal{N} = \text{Ker}(p_\phi)$. We need to show that \mathcal{N} is invertible and that the composite morphism $\theta : f^*\mathcal{N} \rightarrow f^*f_*\mathcal{L} \rightarrow \mathcal{L}$ is an isomorphism. Since these conditions are local over Y and f is finite, we can trivialize \mathcal{L} over affines $V = f^{-1}(U)$ of X , so we may assume that $\mathcal{L} = \mathcal{O}_X$.

Let $\bar{\phi} = (i')^*\phi$ and let

$$\pi_\phi = \pi_1 - \bar{\phi}^{-1} \circ \pi_2 : \bar{f}_*\mathcal{O}_{\Sigma'} \rightarrow \bar{g}_*\mathcal{O}_{\Sigma''}.$$

We claim that we can recover \mathcal{N} from π_ϕ as

$$(1.15) \quad \mathcal{N} = \text{Ker}(i_*\pi_\phi \circ (i')) .$$

Since the diagram

$$\begin{array}{ccc} f_*\mathcal{O}_X & \xrightarrow{p_\phi} & g_*\mathcal{O}_R \\ (i') \downarrow & & \downarrow j \\ i_*\bar{f}_*\mathcal{O}_{\Sigma'} & \xrightarrow{i_*\pi_\phi} & i_*\bar{g}_*\mathcal{O}_{\Sigma''} \end{array}$$

commutes, it is equivalent to prove that $\mathcal{N} = \text{Ker}(j \circ p_\phi)$. Since by definition $\mathcal{N} = \text{Ker}(p_\phi)$, it is sufficient to prove that $\text{Ker}(j) \cap \text{Im}(p_\phi) = \{0\}$. But this is clear since the homomorphism $g_*\mathcal{O}_R \rightarrow f_*\mathcal{O}_X \oplus i_*\bar{g}_*\mathcal{O}_{\Sigma''}$ is injective, by the sequence (1.4) pushed to Y , and since $\Delta \circ p_\phi$ is equal to the zero homomorphism (because $\Delta^*\phi = \text{id}$ by the cocycle condition). Hence the claim is proved.

Since \bar{f} is faithfully flat, hence a strict descent morphism, the descent datum $(\bar{\mathcal{L}}, \bar{\phi})$ is effective; say it descends to $\bar{\mathcal{N}}$, where $\bar{\mathcal{N}}$ is invertible on Σ , and we have an isomorphism $\bar{\theta} : \mathcal{O}_{\Sigma'} \xrightarrow{\sim} \bar{f}^*\bar{\mathcal{N}}$. Let $U = \text{Spec } A$ be an affine open subset of Y , let $V = U \cap \Sigma$, and choose U small enough so that $\bar{\mathcal{N}}|_V \xrightarrow{\sim} \mathcal{O}_V$. Let $U' = f^{-1}(U) = \text{Spec } B$ and $I = \text{Ann}_A(B/A)$ be the annihilator. Let the unit \bar{u} in B/I correspond to the isomorphism $B/I \xrightarrow{\sim} B/I$ induced by the composition $\mathcal{O}_{V'} \xrightarrow{\sim} \bar{f}^*\bar{\mathcal{N}}|_{V'} \xrightarrow{\sim} \mathcal{O}_{V'}$, where $V' = U' \cap \Sigma'$. Then diagram

(1.9) reads

$$\begin{array}{ccc} B/I \otimes_{A/I} B/I & \xrightarrow{\bar{\phi}|_{V''}} & B/I \otimes_{A/I} B/I \\ \bar{u} \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \bar{u} \\ B/I \otimes_{A/I} B/I & \xrightarrow{\text{id}} & B/I \otimes_{A/I} B/I, \end{array}$$

so that $\bar{\phi}|_{V''} = \bar{u} \otimes \bar{u}^{-1}$, where $V'' = V' \times_V V'$. Lift \bar{u} to a unit u in B . Then it is easy to see that the square on the right commutes in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{p_1 - p_2} & B/I \otimes_{A/I} B/I \\ & & & & \downarrow u & & \downarrow \bar{u} \otimes \text{id} \\ 0 & \longrightarrow & \Gamma(U, \mathcal{N}) & \longrightarrow & B & \xrightarrow{p_1 - \bar{\phi}|_{V''} p_2} & B/I \otimes_{A/I} B/I, \end{array}$$

and the horizontal sequences are exact by (1.15). Hence we get an isomorphism $\lambda : A \rightarrow \Gamma(U, \mathcal{N})$, so that \mathcal{N} is invertible on U . Finally, it is easy to see that the following diagram is commutative:

$$\begin{array}{ccccccc} B = A \otimes_A B & \xrightarrow{p_2} & B \otimes_A B & \xrightarrow{\Delta} & B \\ \lambda \otimes \text{id} \downarrow & & u \otimes \text{id} \downarrow & & \downarrow u \\ \Gamma(U, \mathcal{N}) \otimes_A B & \xrightarrow{l} & B \otimes_A B & \xrightarrow{\Delta} & B, \end{array}$$

where l is induced by the inclusion $\Gamma(U, \mathcal{N}) \hookrightarrow B$. The second row of this diagram corresponds to the canonical morphism $\theta : f^* \mathcal{N} \rightarrow f^* f_* \mathcal{O}_X \rightarrow \mathcal{O}_X$ restricted to U' , which we then see to be an isomorphism. Since we can cover Y with open sets U as above, we conclude that the descent datum (\mathcal{L}, ϕ) is effective, so f is a strict descent morphism.

By assumption, $(\sigma_Y)_* \mathcal{O}_Y = \mathcal{O}_S$. Since f is surjective and birational, $(\sigma_X)_* \mathcal{O}_X = (\sigma_Y)_* \mathcal{O}_Y$, so that $\underline{K}_f = \underline{E}\underline{G}_f = \underline{G}_f = \underline{G}_{\bar{f}}$. Also, we see that for any S -scheme T , a line bundle \mathcal{L} on X_T descends to Y_T if and only if $\bar{\mathcal{L}}$ on Σ'_T descends to Σ_T . Hence, the sequence sequence (1.12) is exact in the *fppf* topology, since any element of $\underline{\text{Pic}}_{X/S}(T)$ (resp. $\underline{\text{Pic}}_{Y/S}(T)$) can be represented by a line bundle on $X_{T'}$ (resp. $Y_{T'}$) after a suitable *fppf* base extension $T' \rightarrow T$. Finally, if X has a section, then so does Y and we do not need to apply the *fppf* base change. \square

The functor $\underline{G}_{\bar{f}}$ is a closed subfunctor of the functor

$$T \mapsto \text{Cov}_{\bar{f}}(\mathcal{O}_{\Sigma'_T}) = H^0(\Sigma''_T, \mathcal{O}_{\Sigma''_T}^*).$$

If Σ and Σ' are proper over S , so that Σ'' is also proper over S , then according to [21, Lemme 2.4] the latter functor is representable by an affine scheme over S , so $\underline{G}_{\bar{f}}$ is then

representable by an affine group scheme $G_{\bar{f}}$ over S .

As an important example, suppose $\Sigma = S$ and that $\bar{f} : \Sigma' \rightarrow S$ is faithfully flat and finite of order 2. Let $s \in S$ be a closed point. Then,

$$(1.16) \quad (G_{\bar{f}})_s = \begin{cases} \mathbb{G}_m & \text{if } \Sigma'_s = \text{Spec}(\kappa(s) \oplus \kappa(s)) \quad (\text{case of a node}), \\ \mathbb{G}_a & \text{if } \Sigma'_s = \text{Spec}(\kappa(s)[\epsilon]/(\epsilon^2)) \quad (\text{case of a cusp}). \end{cases}$$

Given a relatively birational morphism $f : X \rightarrow Y$ which does not satisfy the admittedly fairly restrictive conditions of Theorem 1.11, it might still be that there exists an open subset $U \subseteq Y$ such that the restricted morphism $f^{-1}(U) \rightarrow U$ does. Thus, we need to be able to relate $\underline{\text{Pic}}_{Y/S}$ and $\underline{\text{Pic}}_{U/S}$, as in the following lemma. See also [12, Lemma 2.7].

Lemma 1.17. *Let Y be an S -variety whose geometric fibers satisfy Serre's S_2 condition. Let $U \subseteq Y$ be an open subscheme, flat over S , and assume that $\text{Codim}(Y - U) \geq 2$. Then the map of relative Picard functors*

$$\underline{\text{Pic}}_{Y/S} \rightarrow \underline{\text{Pic}}_{U/S}$$

induced by the inclusion is injective. Furthermore, the canonical map $(\sigma_Y)_ \mathcal{O}_Y \rightarrow (\sigma_U)_* \mathcal{O}_U$ is universally an isomorphism.*

PROOF. For any line bundle \mathcal{L} on Y , the canonical map $\rho_{\mathcal{L}} : (\sigma_Y)_* \mathcal{L} \rightarrow (\sigma_U)_*(\mathcal{L}|_U)$ is an isomorphism. Indeed, on the fiber above any $s \in S$, the map $\rho_{\mathcal{L}}$ fits in the long exact sequence of local cohomology:

$$H_{Y_s - U_s}^0(Y_s, \mathcal{L}_s) \longrightarrow H^0(Y_s, \mathcal{L}_s) \longrightarrow H^0(U_s, \mathcal{L}_s) \longrightarrow H_{Y_s - U_s}^0(Y_s, \mathcal{L}_s),$$

whose end terms vanish because Y_s is S_2 , according to [10]. Hence $\rho_{\mathcal{L}}$ is injective and its cokernel is flat on S by [11, 8.11.1]. Since the cokernel vanishes on the fibers, it must vanish everywhere, hence $\rho_{\mathcal{L}}$ is an isomorphism. Thus the second assertion of the lemma holds by putting $\mathcal{L} = \mathcal{O}_Y$, and since the conditions are stable under base change.

Suppose now \mathcal{L} is a line bundle on Y which becomes trivial on U over S , that is, there is an isomorphism $(\sigma_U)^* \mathcal{M} \rightarrow \mathcal{L}|_U$. Tensoring by $(\sigma_U)^* \mathcal{M}^{-1}$, we get an isomorphism $\phi_U : \mathcal{O}_U \rightarrow \mathcal{N}|_U$, where we let $\mathcal{N} = \mathcal{L} \otimes (\sigma_Y)^* \mathcal{M}^{-1}$. Since $\rho_{\mathcal{N}}$ is an isomorphism, there exists an isomorphism $\phi : \mathcal{O}_Y \rightarrow \mathcal{N}$ lifting ϕ_U , and similarly for ϕ_U^{-1} . The compositions $\phi \circ \phi^{-1}$ and $\phi^{-1} \circ \phi$ must be the identity maps because they restrict to the identity maps on U and $\rho_{\mathcal{O}_Y}$ and $\rho_{\mathcal{N}}$ are isomorphisms. Hence \mathcal{L} is isomorphic to $(\sigma_Y)^* \mathcal{M}$. Since the sheafification functor is exact, we get the desired result. \square

2. CURVES, JACOBIANS AND PRESENTATIONS

(2.1) An S -curve is an S -variety C , projective over S , whose geometric fibers are of dimension one. In particular, $\text{Pic}_{C/S} = J_C$ exists for an S -curve C . It is well-known that J_C decomposes as

$$J_C = \coprod_{n \in \mathbb{Z}} J_C^n,$$

where J_C^n parametrizes invertible sheaves \mathcal{L} on C such that \mathcal{L}_s has Euler characteristic n on C_s for any $s \in S$. A *torsion-free, rank 1 sheaf* on a curve C over an algebraically closed field k is a coherent \mathcal{O}_C -module \mathcal{I} generically isomorphic to \mathcal{O}_C and satisfying Serre's S_1 condition. Over an arbitrary base S , a coherent \mathcal{O}_C -module \mathcal{I} is a torsion-free, rank 1 sheaf if it is flat over S and if \mathcal{I}_s is a torsion-free, rank 1 sheaf on the fiber C_s for any geometric point $s \in S$. For an S -curve C , we let \bar{J}_C be the compactified Jacobian of C , which represents the étale sheaf associated to the functor

$$T \mapsto \{\text{isomorphism classes of torsion-free, rank 1 sheaves } \mathcal{I} \text{ on } C_T\}.$$

The scheme \bar{J}_C also decomposes as a union

$$\bar{J}_C = \coprod_{n \in \mathbb{Z}} \bar{J}_C^n.$$

See [2]. Of course, J_C^n lies as an open subset of \bar{J}_C^n . There is an Abel map

$$\mathcal{A}^m : \text{Hilb}_{C/S}^m \rightarrow \bar{J}_C^{1-p-m},$$

which is smooth if the fibers of C are Gorenstein and if $m \geq 2p-1$, where p is the arithmetic genus of C . See again [2] for a more general theory of Abel maps, if C is not Gorenstein. We recall that \bar{J}_C^n is integral if and only if C is locally planar; see [1, Theorem 9] and [16, Theorem 1].

A relatively birational morphism $\pi : C' \rightarrow C$ of S -curves is called a *partial normalization* of C . Since we assume S -curves to be projective and π is necessarily quasi-finite, a partial normalization is automatically finite and surjective. Set $\delta_\pi = \chi(\mathcal{B}(s))$, which is independent of $s \in S$ by flatness; we call δ_π the *genus change* of π . In contrast to the notation for a general relatively birational morphism f , we put $Q_\pi = \text{Supp}(\mathcal{B}_\pi) \subseteq C$ and $Q'_\pi = \pi^{-1}(Q_\pi) \subseteq C'$, and let $\bar{\pi} : Q' \rightarrow Q$ be the restricted morphism. For any partial normalization $\pi : C' \rightarrow C$, there is a canonical closed embedding $\epsilon_\pi : \bar{J}_{C'}^n \hookrightarrow \bar{J}_C^n$ given by push-forward of rank 1, torsion-free sheaves; we will always identify $\bar{J}_{C'}^n$ with its image $\epsilon_\pi(\bar{J}_{C'}^n)$ in \bar{J}_C^n . Again, when there can be no confusion about the morphism π in question, we omit the subscript π .

(2.2) We can apply the results of the previous chapter to partial normalizations of curves.

We immediately obtain the following.

Proposition 2.3. *Let $\pi : C' \rightarrow C$ be a partial normalization of S -curves such that \mathcal{B} is locally free on Q . Then the sequence*

$$(2.4) \quad 0 \longrightarrow G \longrightarrow \mathrm{Pic}_{C/S} \xrightarrow{\pi^*} \mathrm{Pic}_{C'/S} \longrightarrow 0$$

is exact in the fppf topology, where $G = \underline{G}_{\bar{\pi}}$.

PROOF. Since C is proper over S with integral geometric fibers, we have $(\sigma_C)_* \mathcal{O}_C = \mathcal{O}_S$. Hence we can apply Theorem 1.11. It remains to show that $\bar{\pi}^* : \mathrm{Pic}_{Q/S} \rightarrow \mathrm{Pic}_{Q'/S}$ is surjective in the *fppf* topology. This is clear since $\mathrm{Pic}_{Q'/S}$ is trivial. Indeed, any line bundle \mathcal{L} on Q'_T trivializes after a suitable Zariski base change, since Q'_T is finite and flat over T by Lemma 1.2(1). \square

If $\delta = 1$, then $Q \xrightarrow{\sim} S$ and \mathcal{B} must thus be invertible on Q , so we can apply the above proposition. Moreover, $\Sigma' \rightarrow \Sigma$ is faithfully flat and finite of rank 2, so $\underline{G}_{\bar{f}}$ is representable by a scheme G over S and by (1.16) the fibers of G over S are either \mathbb{G}_a or \mathbb{G}_m . Over an algebraically closed field, recall that any partial normalization $\pi : C' \rightarrow C$ can be factored (in potentially many different ways) into a composition

$$C' = C_\delta \longrightarrow C_{\delta-1} \longrightarrow \cdots \longrightarrow C_0 = C$$

with $\delta_{C_r \rightarrow C_{r-1}} = 1$; see [15, Chapter I]. Hence, considering the normalization $\tilde{C} \rightarrow C$ and by induction on the genus deficiency of C , the compactified Jacobian of any integral curve C over k appears as an extension of the Jacobian of \tilde{C} by an affine algebraic group, recovering, or rather paraphrasing, [20, Corollary 4.2].

(2.5) Let $\pi : C' \rightarrow C$ be any partial normalization of S -curves and let T be any S -scheme. Abusing notation throughout, we will denote by $\pi : C'_T \rightarrow C_T$ the morphism induced by π . A *presentation* for π is an injective \mathcal{O}_{C_T} -homomorphism $h : \mathcal{I} \rightarrow \pi_* \mathcal{I}'$, where \mathcal{I} and \mathcal{I}' are torsion-free, rank 1 sheaves on C_T and C'_T respectively, whose cokernel \mathcal{N} is flat over T , has support contained in $Q_T \subseteq C_T$ and is such that $\mathrm{length}(\mathcal{N}(t)) = \delta$ for any $t \in T$. For any torsion-free, rank 1 sheaf \mathcal{I} on C , let \mathcal{I}^π be the torsion-free, rank 1 sheaf on C' generated by $\mathcal{I}|_{C-Q}$. There is a canonical embedding $h_{\mathcal{I}} : \mathcal{I} \rightarrow \pi_* \mathcal{I}^\pi$. By [6, Lemma 3.3], for any presentation $h : \mathcal{I} \rightarrow \pi_* \mathcal{I}'$ there is an embedding $\mathcal{I}^\pi \rightarrow \mathcal{I}'$ on C' such that h factors as

$$\mathcal{I} \xrightarrow{h_{\mathcal{I}}} \pi_* \mathcal{I}^\pi \longrightarrow \pi_* \mathcal{I}'.$$

For a partial normalization $\pi : C' \rightarrow C$ of S -curves, we let Pres_π be the *presentation*

scheme, which represents the étale sheaf associated to the functor

$$T \mapsto \{\text{equivalence classes of presentations } h : \mathcal{I} \rightarrow \pi_* \mathcal{I}' \text{ on } C_T\},$$

where two presentations are considered equivalent if they differ by multiplication by an invertible sheaf. The scheme Pres_π decomposes as a union

$$\text{Pres}_\pi = \coprod_{n \in \mathbb{Z}} \text{Pres}_\pi^n,$$

where a presentation $h : \mathcal{I} \rightarrow \pi_* \mathcal{I}'$ is in Pres_π^n if \mathcal{I} is in \bar{J}_C^n ; see [3]. There are two canonical maps from Pres_π^n , namely

$$\kappa : \text{Pres}_\pi^n \rightarrow \bar{J}_C^n$$

sending a presentation $h : \mathcal{I} \rightarrow \pi_* \mathcal{I}'$ to its source \mathcal{I} , and

$$\kappa' : \text{Pres}_\pi^n \rightarrow \bar{J}_{C'}^{n+\delta}$$

sending h to \mathcal{I}' .

Let $\pi'' : C'' \rightarrow C$ be another partial normalization of C , and define $\text{Pres}_{\pi, C''}^n$ to be the fiber of Pres_π^n over the canonical embedding of $\bar{J}_{C''}^n$ inside \bar{J}_C^n , that is, the product

$$\begin{array}{ccc} \text{Pres}_{\pi, C''}^n & \hookrightarrow & \text{Pres}_\pi^n \\ \downarrow & & \downarrow \\ \bar{J}_{C''}^n & \xrightarrow{\epsilon''} & \bar{J}_C^n. \end{array}$$

In particular, we have $\text{Pres}_{\pi, C}^n = \text{Pres}_\pi^n$. The canonical maps κ and κ' from Pres_π^n restrict to maps $\text{Pres}_{\pi, C''}^n \rightarrow \bar{J}_{C''}^n$ and $\text{Pres}_{\pi, C''}^n \rightarrow \bar{J}_{C'}^n$, which, abusing notation, we also denote by κ and κ' respectively. It is in general difficult to describe $\text{Pres}_{\pi, C''}^n$ adequately. However, we can make some progress if π factors through π'' , that is, if π'' is an *intermediate* partial normalization. Indeed, let $\pi' : C' \rightarrow C''$ be the morphism through which π factors, so that we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{\pi} & C' \\ & \searrow \pi' & \nearrow \pi'' \\ & C'' & \end{array}$$

Let $Q'' = (\pi'')^{-1}(Q)$. Assume there is a universal sheaf \mathbf{I}' on $C' \times \bar{J}_{C'}^{n+\delta}$ (which is the case over an algebraically closed field, for example), and let the sheaf $\mathcal{F}_{C''}$ on $Q'' \times_S \bar{J}_{C'}^{n+\delta}$ be defined by

$$\mathcal{F}_{C''} = (\pi'_* \mathbf{I}')|_{(Q'' \times_S \bar{J}_{C'})}.$$

Then the proof of [3, Proposition 9] can be easily adapted to show there is a canonical isomorphism of schemes over $\bar{J}_{C'}^{n+\delta}$:

$$\text{Pres}_{\pi, C''}^n = \text{Quot}_{\mathcal{F}_{C''}/Q'' \times_S \bar{J}_{C'}^{n+\delta}/\bar{J}_{C'}^{n+\delta}}^\delta.$$

We then have the following result.

Proposition 2.6. *Assume that Q' lies in the regular locus of C' and that π factors through π'' . Then the map $\kappa' : \text{Pres}_{\pi, C''}^n \rightarrow \bar{J}_{C'}^{n+\delta}$ makes $\text{Pres}_{\pi, C''}^n$ into a bundle over $\bar{J}_{C'}^{n+\delta}$ with fiber*

$$F_{\pi, C''} = \text{Quot}_{\bar{\pi}'_* \mathcal{O}_{Q'}/Q''/S}^\delta,$$

where $\bar{\pi}' : Q' \rightarrow Q''$ is the restricted morphism. In particular, if Q'' has fibers of length one over S , then

$$F_{\pi, C''} = \text{Grass}_S^\delta((\sigma_{Q'})_* \mathcal{O}_{Q'}).$$

PROOF. For any open subset $U \subseteq \bar{J}_{C'}^{n+\delta}$, consider the diagram

$$\begin{array}{ccccc} Q' \times_S U & \xrightarrow{g'} & Q' \times_S \bar{J}_{C'}^{n+\delta} & \xrightarrow{q'} & C' \times_S \bar{J}_{C'}^{n+\delta} \\ \pi' \downarrow & & \pi' \downarrow & & \downarrow \pi' \\ Q'' \times_S U & \xrightarrow{g''} & Q'' \times_S \bar{J}_{C'}^{n+\delta} & \xrightarrow{q''} & C'' \times_S \bar{J}_{C'}^{n+\delta} \end{array}$$

Since the formation of the Quot-scheme commutes with base change, we have

$$\text{Quot}_{\mathcal{F}_{C''}/Q'' \times_S \bar{J}_{C'}^{n+\delta}/\bar{J}_{C'}^{n+\delta} \times \bar{J}_{C'}^{n+\delta} U}^\delta = \text{Quot}_{(g'')^* \mathcal{F}_{C''}/Q'' \times_S U/U}^\delta.$$

Since Q' lies in the regular locus of C' , the sheaf $(q')^* \mathbf{I}'$ is invertible on $Q' \times_S \bar{J}_{C'}^{n+\delta}$. Since Q' has finite length over S , we can choose the subset U of $\bar{J}_{C'}^{n+\delta}$ small enough so that $(q')^* \mathbf{I}'|_{Q' \times_S U} \simeq \mathcal{O}_{Q' \times_S U}$. Since π' is affine, the formation of $\mathcal{F}_{C''}$ commutes with base change, so that $(g'')^* \mathcal{F}_{C''} = \bar{\pi}'_*(g')^*(q')^* \mathbf{I}' \simeq \bar{\pi}'_* \mathcal{O}_{Q' \times_S U}$, hence

$$\text{Quot}_{(g'')^* \mathcal{F}_{C''}/Q'' \times_S U/U}^\delta = \text{Quot}_{\bar{\pi}'_* \mathcal{O}_{Q' \times_S U}/Q'' \times_S U/U}^\delta.$$

But clearly $\bar{\pi}'_* \mathcal{O}_{Q' \times_S U} = p^* \bar{\pi}'_* \mathcal{O}_{Q'}$, where $p : Q'' \times_S U \rightarrow Q''$ is the projection, so that

$$\text{Quot}_{\bar{\pi}'_* \mathcal{O}_{Q' \times_S U}/Q'' \times_S U/U}^\delta = \text{Quot}_{p^* \bar{\pi}'_* \mathcal{O}_{Q'}/Q'' \times_S U/U}^\delta = \text{Quot}_{\bar{\pi}'_* \mathcal{O}_{Q'}/Q''/S}^\delta \times_S U,$$

the latter equality holding since, once again, the formation of Quot commutes with base change. \square

Put $C'' = C$ and assume that $\delta = 1$ in the above proposition. Then the fibers of Q' have

length 2 over S , so $F_{\pi,C}$ is a twisted \mathbb{P}_S^1 and we immediately recover [3, Theorem 10]. On the other hand, there is an extensive bestiary of examples of $F_{\pi,C''}$ for more general δ and C'' , ranging from smooth to outright pathological. We will content ourselves here with two simple ones, assuming S is the spectrum of an algebraically closed field k and that $C'' = C$. Then Q and Q' are Artin schemes over k . Hence $F_{\pi,C}$ embeds as the closed subscheme of $\text{Grass}_k^\delta(H^0(Q', \mathcal{O}_{Q'}))$ parametrizing k -subspaces of $H^0(Q', \mathcal{O}_{Q'})$ which are $H^0(Q, \mathcal{O}_Q)$ -modules. Carefully choosing bases of the latter two vector spaces, it is possible to find the equations defining $F_{\pi,C}$ in the distinguished affine open subsets of the Grassmanian through the Plücker embedding of the latter in a projective space. For instance, if C' is smooth and C has two nodes, then

$$Q = \text{Spec } k \times k, \quad Q' = \text{Spec}(k \times k) \times (k \times k),$$

and we find that $F_{\pi,C} \subseteq \text{Grass}_k^2(4) \subseteq \mathbb{P}^5$ is isomorphic to the disjoint union of $\mathbb{P}^1 \times \mathbb{P}^1$ and two points. If C' is smooth and C has a single tacnode, then

$$Q = \text{Spec } k[\epsilon]/(\epsilon^2), \quad Q' = \text{Spec } k[\epsilon]/(\epsilon^2) \times k[\epsilon]/(\epsilon^2),$$

and the reduction of $F_{\pi,C}$ is isomorphic to the quadric cone in \mathbb{P}^3 , but there is a rather intricate embedded point at the vertex of this cone, which brings the dimension of the tangent space at the vertex to 4.

The above examples suggest that the schemes $F_{\pi,C''}$ over an algebraically closed field are rationally connected, or at least have rationally connected components. This is indeed the case: the proof of [23] can be adapted to show that the Quot-scheme $\text{Quot}_{M/A/k}^d$ of a module M of finite length over an Artin ring A has rationally connected components. Whether these components (or, more precisely, their reductions) are unirational or rational is an open question. Note that if A is local, then the proof of [7, Proposition 2.2] is easily generalized to show that the scheme $\text{Quot}_{M/A/k}^d$ is connected.

(2.7) Let C be an S -curve and let $\pi : C' \rightarrow C$ be any partial normalization with genus change one. There is then a canonical map $\sigma : \text{Pres}_{\pi,C'}^n \rightarrow Q'$, defined as follows. A T -point of $\text{Pres}_{\pi,C'}^n$ corresponds to a presentation $h : \pi_* \mathcal{I} \rightarrow \pi_* \mathcal{L}$, which in turn corresponds to an embedding $\mathcal{I} \hookrightarrow \mathcal{L}$ on C'_T whose cokernel \mathcal{N} is flat and has length one over T . Hence the support Z of \mathcal{N} , which lies in Q'_T by definition, is flat and has length one over T , so the structure morphism $Z \rightarrow T$ is an isomorphism. There is thus a morphism $T \rightarrow Q'_T$ having image Z , and composing with the projection Q'_T , we obtain a T -point of Q' .

Assume in addition that Q' lies in the regular locus of C' . Define the isomorphism

$$\lambda : \bar{J}_{C'}^{n+1} \times_S Q' \xrightarrow{\sim} \bar{J}_{C'}^n \times_S Q'$$

by mapping a pair (\mathcal{L}, q') to $(\mathcal{M}_{q'}\mathcal{L}, q')$, where $\mathcal{M}_{q'}$ is the ideal sheaf of q' on C' . Then the following diagram commutes and all the morphisms in it are isomorphisms:

$$(2.8) \quad \begin{array}{ccc} & \text{Pres}_{\pi, C'}^n & \\ (\kappa', \sigma) \swarrow & & \searrow (\kappa, \sigma) \\ \bar{J}_{C'}^{n+1} \times_S Q' & \xrightarrow{\lambda} & \bar{J}_{C'}^n \times_S Q'. \end{array}$$

Indeed, both the commutativity and the fact that (κ', σ) is an isomorphism follow immediately from [3, Theorem 16], if we notice that the map $\epsilon' : \bar{J}_{C'}^{n+1} \times_S Q' \rightarrow \text{Pres}_{\pi}^n$ defined there by mapping a pair (\mathcal{L}, q') to the presentation $\pi_*\mathcal{M}_{q'}\mathcal{L} \rightarrow \pi_*\mathcal{L}$ inverts (κ', σ) . In particular, we see from diagram (2.8) that the morphism $\kappa : \text{Pres}_{\pi, C'}^n \rightarrow \bar{J}_{C'}^n$ is isomorphic to the projection $\bar{J}_{C'}^n \times_S Q' \rightarrow \bar{J}_{C'}^n$, under the identification provided by (κ, σ) .

(2.9) We will work for the remainder of this Section over an algebraically closed field k . For any partial normalization $\pi : C' \rightarrow C$, we introduce stratifications $\mathcal{S}_{\pi}^{\cdot, n}$ and $\mathcal{T}_{\pi}^{\cdot, n}$ of $\bar{J}_{C'}^n$ and Pres_{π}^n respectively as follows. Let $\mathcal{S}_{\pi}^{0, n}$ denote the open subset of $\bar{J}_{C'}^n$, consisting of torsion-free, rank 1 sheaves on C' invertible at Q . We then define $\mathcal{S}_{\pi}^{i, n}$ recursively by setting

$$\mathcal{S}_{\pi}^{i+1, n} = \text{Im} [Z_{\pi}^{i, n+1} \rightarrow \bar{J}_{C'}^n],$$

where $Z_{\pi}^{i, n+1}$ is the open subset of $C \times_S \mathcal{S}_{\pi}^{i, n+1}$ consisting of pairs (p, \mathcal{I}) where \mathcal{I} is invertible at p , and $Z_{\pi}^{i, n+1} \rightarrow \bar{J}_{C'}^n$, maps a pair (p, \mathcal{I}) to $\mathcal{M}_p\mathcal{I}$. We let $\mathcal{T}_{\pi}^{i, n} = \kappa^{-1}(\mathcal{S}_{\pi}^{i, n})$. To simplify notation, we omit the superscript n , as it will always be obvious. We also set $\mathcal{S}^i = \mathcal{S}_{\tilde{\pi}}^i$, where $\tilde{\pi} : \tilde{C} \rightarrow C$ is the normalization, and $\mathcal{T}^i = \mathcal{T}_{\tilde{\pi}}^i$. It is clear by induction that $\mathcal{S}_{\pi}^i \subseteq \mathcal{S}_{\pi}^{i+1}$, hence we obtain stratifications

$$\begin{aligned} J_C^n = \mathcal{S}^0 &\subseteq \mathcal{S}_{\pi}^0 \subseteq \mathcal{S}_{\pi}^1 \subseteq \cdots \subseteq \bar{J}_C^n \\ \text{Pres}_{\pi}^n \cap \mathcal{T}^0 &\subseteq \mathcal{T}_{\pi}^0 \subseteq \mathcal{T}_{\pi}^1 \subseteq \cdots \subseteq \text{Pres}_{\pi}^n. \end{aligned}$$

We will only be using the $i = 0, 1$ steps of these stratifications. When no confusion about π can arise, we denote by $\kappa^1 : \mathcal{T}_{\pi}^1 \rightarrow \mathcal{S}_{\pi}^1$ the restricted morphism induced by κ .

(2.10) If $\pi : C' \rightarrow C$ is a partial normalization and P a point of C , we say that π is a partial normalization *at* P if Q is supported at P . Any curve C has a unique *normalization* $\tilde{\pi} : \tilde{C} \rightarrow C$, where \tilde{C} is regular. We define the *genus deficiency* of C to be $\delta_{\tilde{\pi}}$; it is clearly equal to $p - g$, where p and g are the arithmetic and geometric genera of C respectively. There are also other canonical partial normalizations associated to C , which we now describe. Suppose P is a singular point of C . The blow-up $\pi_P : C_P \rightarrow C$ of C along the ideal sheaf \mathcal{M}_P of P is a partial normalization. We can continue to blow-up singular points if necessary to construct the *desingularization* $\pi_{(P)} : C_{(P)} \rightarrow C$ of C at P , which is such that $\pi_{(P)}^{-1}(P)$ lies in the regular locus of $C_{(P)}$ and $Q_{\pi_{(P)}}$ is supported at P . If C is locally planar,

then both C_P and $C_{(P)}$ are also locally planar, but the genus changes of π_P and $\pi_{(P)}$ are in general unpredictable. On the other hand, factoring the blow-up $C_P \rightarrow C$ into steps with genus change one, we see that there always exists at least one partial normalization at P with genus change one. Moreover, if C is Gorenstein, this partial normalization with genus change one is unique; we denote it by $\pi_{[P]} : C_{[P]} \rightarrow C$. Indeed, the conductor of \mathcal{O}_C in $\pi_{[P]*}\mathcal{O}_{C_{[P]}}$ must then be equal to the maximal ideal sheaf \mathcal{M}_P of P and, conversely, $\pi_{[P]*}\mathcal{O}_{C_{[P]}} = \mathcal{H}om_C(\mathcal{M}_P, \mathcal{O}_C)$ is determined by \mathcal{M}_P . Hence, a Gorenstein curve has as many partial normalizations with genus change one as it has singularities, and any partial normalization factors through at least one of them. It is important to note that $C_{[P]}$ is in general not Gorenstein, though.

3. DOUBLE POINTS

In this Section we work over an algebraically closed field k . A point P of an integral curve C is a *double point* if C has multiplicity 2 at that point. We have a local characterization of double points as follows.

Proposition 3.1. *Let C be an integral curve and let P be a point of C such that C is locally planar at P . Then P is a double point of C if and only if one of the following equivalent conditions is satisfied:*

1. $\mathcal{O}_{C,P}$ is analytically isomorphic to $k[[x, y]]/(x^2 + f(y)x + g(y))$ for some $f(y), g(y) \in k[[y]]$. If $\text{char } k \neq 2$, then we can complete the square so that $\mathcal{O}_{C,P}$ is analytically isomorphic to $k[[x, y]]/(x^2 - y^r)$ for some integer r .
2. $\dim_k I/\mathfrak{m}_P I \leq 2$ for every ideal $I \subseteq \mathcal{O}_{C,P}$.
3. $\dim_k \mathfrak{m}_P^2/\mathfrak{m}_P^3 = 2$, where \mathfrak{m}_P is the maximal ideal of $\mathcal{O}_{C,P}$.
4. The blow-up C_P of C at P has genus change one.
5. The partial normalization $C_{[P]}$ of C at P with genus change one has at worst double points over P .

Furthermore, any of the above condition implies the following:

6. If $\pi : C' \rightarrow C$ is the blow-up of C at P , then the sheaf \mathcal{B} is invertible on Q .

PROOF. We let (0) stand for the condition that P be a double point of C . Suppose (0) holds. Since k is infinite, by [24, VIII, Theorem 22] there exists a non zero divisor $y \in \mathcal{O}_{C,P}$ such that the multiplicity of (y) in $\mathcal{O}_{C,P}$ is equal to 2. Since $\mathcal{O}_{C,P}$ is Cohen-Macaulay, we have an isomorphism

$$\mathcal{O}_{C,P}/(y)[t] \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} (y^n)/(y^{n+1}).$$

Thus by [24, VIII, Theorem 23], we have that $\text{length } \mathcal{O}_{C,P}/(y) = 2$. Completing, we must have that $\hat{\mathcal{O}}_{C,P}$ is free of rank 2 over $k[[y]]$, which proves (1). The converse is immediate. It is sufficient to prove (1) \Rightarrow (2) on the completion. Any ideal $I \subseteq \hat{\mathcal{O}}_{C,P}$ is free of rank 2 over $k[[y]]$. If I is generated by two elements over $k[[y]]$, then it is also generated by the same elements over $\hat{\mathcal{O}}_{C,P}$, hence (2) holds. The implication (2) \Rightarrow (3) is immediate. We now prove that (3) \Rightarrow (1). Since C is locally planar, $\hat{\mathcal{O}}_{C,P} \simeq k[[x,y]]/(f)$ for some f in \mathfrak{m}_P^2 . But (3) implies that f cannot be in \mathfrak{m}_P^3 . Hence we get (1). The equivalence of (0) and (4) follows immediately from the formula in [13, Corollary V.3.7]. Moreover, if (4) holds, then the partial normalization $\pi_{[P]}$ with genus change one at P must coincide with π_P , and $C_{[P]} = C_P$ must have at worst double points over P since the multiplicity at a point P' of C' can not be greater than that of C at $P = \pi_P(P')$, so (5) holds. Conversely, consider $Q' \subseteq C'$. If Q' lies in the regular locus of C' , then P must be a node or a cusp on C . If Q' does not lie in the regular locus, then it must be supported at a point P' of C' , which is assumed to be a double point. Then the maximal ideal \mathfrak{m}_P of $\mathcal{O}_{C,P}$ is an ideal of $\mathcal{O}_{C',P'}$ also since it is the conductor. Also, $\mathfrak{m}_{P'}\mathfrak{m}_P^2 = \mathfrak{m}_P^3$. Thus by (2), we have $\dim_k \mathfrak{m}_P^2/\mathfrak{m}_P^3 = \dim_k \mathfrak{m}_P^2/\mathfrak{m}_{P'}\mathfrak{m}_P^2 \leq 2$, so we conclude by (3) that P is a double point. Finally, (4) trivially implies (6), since the sheaf \mathcal{B} has length one over k and is supported on $Q = \text{Spec } k$. \square

It is important to insist that if P is a double point of C , then the blow-up C_P of C at P is the same as the partial normalization $C_{[P]}$ with genus change one at P . It is not clear whether a curve C has a double point at P as soon as $C_{[P]}$ is only assumed to be locally planar above P .

(3.2) It is easy to actually construct double points, as we now show. The first step is to construct nodes and cusps. If D is an effective divisor of degree 2 on a curve C' supported in the regular locus, then there exists a unique curve C and a morphism $\pi : C' \rightarrow C$ contracting D to a singular point P of C , which is a node or a cusp depending on whether D is supported at a single point or not. It suffices to take C as the curve associated to the *module* D , in the sense of [22, IV 4]. If P' is a double point on C' , we can also increase its complexity. We will construct a map $\pi : C' \rightarrow C$ with genus change one such that C is locally planar at $P = \pi(P')$. By Proposition 3.1(5) above, P will thus be a double point of C . Let the maximal ideal $\mathfrak{m}_{P'} \subseteq \mathcal{O}_{C',P'}$ be generated by x' and y' , where we choose y' such that the multiplicity of (y') in $\mathcal{O}_{C',P'}$ is equal to 2, as in the proof above. For any $[a : b] \in \mathbb{P}_k^1$, let \mathcal{O}' be the subring $k \oplus (ax' + by')$ of $\mathcal{O}_{C',P'}$. Then \mathcal{O}' fits in the inclusions

$$k \oplus \mathfrak{m}_{P'}^2 \subseteq \mathcal{O}' \subseteq \mathcal{O}_{C',P'}$$

hence, according to [15, Proposition 1.1.1], defines a curve C and a homeomorphism $\pi : C' \rightarrow C$ such that π is an isomorphism over $C - P$, where $P = \pi(P')$, and such that $\mathcal{O}_{C,P} = \mathcal{O}'$.

It is clear that π has genus change one. Conversely, any such partial normalization defines a subring $\mathcal{O}' = \mathcal{O}_{C,P}$ of $\mathcal{O}_{C',P'}$ fitting in the inclusions above. To verify whether the curves thus defined are locally planar at P , it suffices to work on the completion. To simplify the discussion, we will assume that $\text{char } k \neq 2$. Then $\hat{\mathcal{O}}_{C',P'} \simeq k[[x', y']]/(x'^2 - y'^r)$. Let \mathfrak{m}' be the maximal ideal of $\hat{\mathcal{O}}'$, so that $\mathfrak{m}' = \mathfrak{m}_{P'}^2 + (ax' + by')$. Suppose $r \geq 2$. If $b = 0$, then $\mathfrak{m}' = (x', y'^2)$ and it is easy to see that $\mathfrak{m}'/\mathfrak{m}'^2$ has dimension 3, so that the curve C defined by \mathcal{O}' is not locally planar at P . On the other hand, if $b \neq 0$, then C will be locally planar at P . If $r = 2$, then C is locally planar at P for any choice of $[a : b]$ in \mathbb{P}_k^1 . If the normalization \tilde{C} of C' has no automorphism, then different choices of $[a : b]$ in \mathbb{P}_k^1 give rise to non-isomorphic curves since the rings \mathcal{O}' are themselves non-isomorphic because their common fraction field has no non-trivial k -automorphism. Hence we have defined at least one curve C with double points to which C' maps, and in most cases, a whole family of such.

In contrast with the local characterization above of double points, we also have the following important global characterizations of curves all of whose singularities are double points.

Proposition 3.3. *Let C be an integral locally planar curve. Then all the singularities of C are double points if and only if one of the following equivalent conditions hold:*

1. *All the proper partial normalizations of C have double points.*
2. *All the proper partial normalizations of C are locally planar.*

Furthermore, any of the above conditions implies the following:

3. *For any partial normalization $\pi : C' \rightarrow C$ at a point P , we have $Q = \text{Spec } k[y]/(y^\delta)$.*
4. *For any partial normalization $\pi : C' \rightarrow C$ of C , the sheaf \mathcal{B} is invertible on Q .*

PROOF. Again we let (0) stand for the condition that all the singularities of C be double points. Then (0) implies (1) by Proposition 3.1(5) above and induction. The implication (1) \Rightarrow (2) is immediate. On the other hand, if C has a point P of multiplicity greater than 2, some partial normalization of C at P is not locally planar. Indeed, the proof of [15, Lemma 1.2.1] shows there exists a partial normalization $C_{\{P\}}$ such that $C_{\{P\}} \rightarrow C$ is a homeomorphism and the singular point P on $C_{\{P\}}$ is a module type. But a singularity of module type of multiplicity greater than 2 cannot be Gorenstein, let alone locally planar. Hence (2) \Rightarrow (0).

We prove that (0) implies (3) and (4) simultaneously. Note that (3) implies (4) because \mathcal{B} has length δ and has support equal to Q . We make the following trivial observation. Let A be a local Artin ring of dimension δ over k , and suppose there is an surjective algebra

homomorphism $A \rightarrow k[\epsilon]/(\epsilon^{\delta-1})$ such that for any d with $0 < d < \delta$, composing the above surjection with the canonical map $k[\epsilon]/(\epsilon^{\delta-1}) \rightarrow k[\epsilon]/(\epsilon^{\delta-d})$ yields an exact sequence of A -modules:

$$0 \longrightarrow k[\epsilon]/(\epsilon^d) \longrightarrow A \longrightarrow k[\epsilon]/(\epsilon^{\delta-d}) \longrightarrow 0.$$

Then A is isomorphic to $k[\epsilon]/(\epsilon^\delta)$. To apply this observation to the present case, given $\pi : C' \rightarrow C$ with genus change δ , note first that since Q is supported at a point, $A = H^0(Q, \mathcal{O}_Q)$ is local. We will use induction on δ . If $\delta = 1$, then Q has length one over k so the result is automatic. Otherwise, assume the result holds for any partial normalization of curves with double points with genus change less than δ . For any d such that $0 < d < \delta$, there exists a unique intermediate partial normalization $\pi'' : C'' \rightarrow C$ with genus change d (the composition of d blow-ups). Let $\pi' : C' \rightarrow C''$ be the morphism through which π factors. We get an exact sequence

$$0 \longrightarrow B_{\pi''} \longrightarrow B_\pi \longrightarrow B_{\pi'} \longrightarrow 0.$$

Hence, using the induction hypothesis and the fact that (4) holds for π'' and π' , we conclude by the above observation. \square

If C has only double points, then \bar{J}_C^n is explicitly stratified by the invertible sheaves on the finite set of its partial normalizations, as expressed in the following Proposition.

Proposition 3.4. *Let all the singularities of C be double points. Then, set-theoretically, \bar{J}_C^n decomposes as*

$$\bar{J}_C^n = \coprod_{C' \rightarrow C} J_{C'}^n,$$

where the union is taken over all curves mapping to C (including C).

PROOF. To prove the first decomposition, we proceed by induction on the genus deficiency δ of C . If $\delta = 0$, then C is smooth and there is nothing to prove. Otherwise, let \mathcal{I} be a non-invertible, torsion-free, rank 1 sheaf on C , and suppose \mathcal{I} is not invertible at a point P . Let $\pi : C' \rightarrow C$ be the blow-up of C at P . Let $R = \mathcal{O}_{C,P}$, $R' = (\pi_* \mathcal{O}_{C'})_P$ and $I = \mathcal{I}_P$. By [2, Lemma 3.3], we can identify I with an ideal of R . Consider the inclusions

$$\mathfrak{m}_P I \subseteq I \subseteq R' I,$$

where \mathfrak{m}_P is the maximal ideal of R . Since R is Gorenstein, \mathfrak{m}_P is equal to the conductor of R in R' . Hence $\mathfrak{m}_P R' I = \mathfrak{m}_P I$. By Proposition 3.1(2), $\dim_k R' I / \mathfrak{m}_P I \leq 2$. By Nakayama's lemma, $\mathfrak{m}_P I \neq I$. In fact, $\dim_k I / \mathfrak{m}_P I > 1$, since \mathcal{I} is not invertible at P . Hence $\dim_k R' I / I = 0$, that is, $I = R' I$. Hence \mathcal{I} is an $\mathcal{O}_{C'}$ -module, that is, $\mathcal{I} = \pi_* \mathcal{I}'$ for

some torsion-free, rank 1 sheaf \mathcal{I}' on C' . By the induction hypothesis, $\mathcal{I}' = \pi_*'' \mathcal{I}''$ for some invertible sheaf \mathcal{I}'' on some partial normalization C'' of C' . Hence the first decomposition holds. \square

Since a curve C with double points is locally planar, the singular locus of Hilb_C^m is equal to the locus of non-invertible sheaves by [4, Proposition 2.3]. Hence, since the Abel map is smooth for large enough m , the singular locus of \bar{J}_C^n is equal to $\bar{J}_C^n - J_C^n$. But by the above Proposition,

$$\bar{J}_C^n - J_C^n = \bigcup_{\substack{C' \rightarrow C \\ \delta=1}} \bar{J}_{C'}^n.$$

Hence the singular locus of \bar{J}_C^n decomposes in as many irreducible components as there are double points on C , each component being isomorphic to the compactified Jacobian of the corresponding partial normalization.

Let $\pi : C' \rightarrow C$ be a partial normalization with genus change one, that is, the blow-up of C at some double point P . We can also extract from the above decomposition some information about the stratification \mathcal{S}_π^1 of \bar{J}_C^n . First note that if Q' lies in the regular locus of C' , that is, if π forms either a node or a cusp, then it is easy to see that \mathcal{S}_π^1 is in fact equal to the whole of \bar{J}_C^n . Also, in general, we have that

$$(3.5) \quad \mathcal{S}_\pi^1 - \mathcal{S}_\pi^0 = \bar{J}_{C'}^n \cap \mathcal{S}_\pi^1.$$

Finally, since

$$\coprod_{\substack{C' \rightarrow C \\ \delta \leq 1}} J_{C'}^n \subseteq \mathcal{S}_\pi^1,$$

we obtain the very useful inequality

$$(3.6) \quad \text{Codim}(\bar{J}_{C'}^n - \mathcal{S}_\pi^1) \geq 2.$$

In Proposition 3.9 and Corollary 3.13 below, we show that the conditions necessary to apply the results of Section 1 are satisfied by the restriction of κ to the open subset \mathcal{T}_π^1 of Pres_π . Proposition 3.9 and its proof are directly inspired by [3, Proposition 17], but the preliminary set-up is more extensive. We first need the following technical lemma.

Lemma 3.7. *Let $\pi : C' \rightarrow C$ be a partial normalization, and suppose C has only double points. Let $h : \mathcal{I} \rightarrow \pi_* \mathcal{I}'$ be a presentation lying in \mathcal{T}_π^1 . Then the cokernel of h is invertible on Q .*

PROOF. The assertion is local on Q , hence we may assume that Q is supported at a double point P , that is, that π is a partial normalization at P . If \mathcal{I} is invertible along Q , then we may assume $\mathcal{I} = \mathcal{O}_C$ since, once again, the assertion is local, and then the result is

simply Proposition 3.3(4). Otherwise, by Proposition 3.4, \mathcal{I} must be equal to $(\pi_P)_*\mathcal{I}^P$ for some invertible sheaf \mathcal{I}^P on the blow-up $\pi_P : C_P \rightarrow C$ of C at P . We may assume that $\mathcal{I}^P = \mathcal{O}_{C_P}$. We prove the result by induction on δ . If $\delta = 1$, the result is clear. Otherwise, for any d with $0 < d < \delta$, let $\pi^{(d)} : C^{(d)} \rightarrow C$ be the partial normalization of C consisting of $d + 1$ blow-ups. Then h factors as

$$(\pi_P)_*\mathcal{O}_{C_P} \rightarrow \pi_*^{(d)}\mathcal{O}_{C^{(d)}} \rightarrow \pi_*\mathcal{I}.$$

The cokernel of the first map above is equal to $k[\epsilon]/(\epsilon^d)$ by Proposition 3.3(3). The cokernel of $\pi^{(d)}\mathcal{O}_{C^{(d)}} \rightarrow \pi_*\mathcal{I}$ is equal to the cokernel of $\mathcal{O}_{C^{(d)}} \rightarrow \pi_*^{C' \rightarrow C^{(d)}}\mathcal{I}'$, which in turn is equal to the cokernel of $\pi_*^{C^{(d)} \rightarrow C^{(d-1)}}\mathcal{O}_{C^{(d)}} \rightarrow \pi_*^{C' \rightarrow C^{(d-1)}}\mathcal{I}'$. The latter is a presentation for the map $C' \rightarrow C^{(d-1)}$ which lies in $\text{Pres}_{\pi_{C' \rightarrow C^{(d-1)}}}^n \cap \mathcal{T}_{\pi_{C' \rightarrow C^{(d-1)}}}^1$, so that by the induction hypothesis and by Proposition 3.3(3), its cokernel is isomorphic to $k[\epsilon]/(\epsilon^{\delta-d})$. Hence we have an exact sequence

$$0 \rightarrow k[\epsilon]/(\epsilon^{\delta-d}) \rightarrow \text{Coker } h \rightarrow k[\epsilon]/(\epsilon^d) \rightarrow 0,$$

and we conclude as in the proof of the implication (0) \Rightarrow (3) in Proposition 3.3. \square

(3.8) To simplify notation, for any m , we set $\mathbf{H}^m = \text{Hilb}_C^m$ and $\mathbf{H}'^m = \text{Hilb}_{C'}^m$. We also define \mathbf{H}_Q^m to be the open subset of \mathbf{H}^m parametrizing subschemes of C supported on $C - Q$, and $\mathbf{H}'_Q{}^m$ to be the open subset of \mathbf{H}'^m parametrizing open subschemes of C' supported on $C' - Q'$. Let $\text{Supp}(Q) = \{P_1, \dots, P_r\}$ and $\pi^{-1}(P_i) = \{P_i^1, \dots, P_i^{j_i}\}$. Let Γ be the open subset of \mathbf{H}^δ parametrizing subschemes of C whose length at P_i is at most one for each i , and Γ' be the open subset of \mathbf{H}'^δ parametrizing subschemes of C' whose length at P_i^j is at most one for at most one j for each i . We have open embeddings $\mathbf{H}'_Q{}^\delta \subseteq \Gamma'$ and $\mathbf{H}_Q^\delta \subseteq \Gamma$, and we let $\gamma' = \Gamma' - \mathbf{H}'_Q{}^\delta$ and $\gamma = \Gamma - \mathbf{H}_Q^\delta$ be the closed complements with the reduced scheme structure. For instance, if $\delta = 1$, which is the only case we will use in the proof of autoduality, then $\Gamma = C$ and $\Gamma' = C'$, with $\gamma = Q$ and $\gamma' = Q'$. If $\delta > 1$, however, Γ' and Γ are not necessarily proper. We claim, however, that the canonical isomorphism $\mathbf{H}'_Q{}^\delta \xrightarrow{\sim} \mathbf{H}_Q^\delta$ extends to a morphism $\Pi : \Gamma' \rightarrow \Gamma$. Let V' and V be the open subsets of $C' \times \mathbf{H}'_Q{}^{\delta-1}$ and $C \times \mathbf{H}_Q^{\delta-1}$ respectively, consisting of pairs (P, Y) such that P is not in the support of Y . We have canonical maps $\beta' : V' \rightarrow \Gamma'$ and $\beta : V \rightarrow \Gamma$, which are étale by the infinitesimal criterion. Let the images of these maps be denoted by W' and W respectively. The fiber product $V' \times_{W'} V$ lies in $V' \times V$, which is itself an open subset of $C' \times \mathbf{H}'_Q{}^{\delta-1} \times C \times \mathbf{H}_Q^{\delta-1}$, and similarly for $V \times_W V$. The canonical map $C' \times \mathbf{H}'_Q{}^{\delta-1} \times C \times \mathbf{H}_Q^{\delta-1} \rightarrow C \times \mathbf{H}_Q^{\delta-1} \times C \times \mathbf{H}_Q^{\delta-1}$ is easily seen to restrict to a map $V' \times_{W'} V \rightarrow V \times_W V$ which makes the following diagram

Cartesian, for $i = 1, 2$:

$$\begin{array}{ccc} V' \times_{W'} V' & \xrightarrow{p_i} & V' \\ \downarrow & & \downarrow \\ V \times_W V & \xrightarrow{p_i} & V. \end{array}$$

Thus we get descent data for the affine scheme V' over V through the morphism $V \rightarrow W$. Since $V \rightarrow W$ is faithfully flat, hence a strict descent morphism, V' descends to a scheme W^* . But since $V' \times_{W'} V'$ and $V' \times_{W^*} V'$ coincide by construction and both $V' \rightarrow W'$ and $V' \rightarrow W^*$ are faithfully flat, the schemes W' and W^* must coincide. In other words, we have constructed a morphism $W' \rightarrow W$. It is clear that W' and W contain the boundaries γ' and γ respectively, and that the morphism $W' \rightarrow W$ just defined coincides with the canonical isomorphism $\mathbf{H}'_{Q'} \rightarrow \mathbf{H}_Q^\delta$ where they are both defined. Hence they glue to give the desired morphism $\Pi : \Gamma' \rightarrow \Gamma$.

Proposition 3.9. *Let the curve C have only double points and let $\pi : C' \rightarrow C$ be a partial normalization. Then in the smooth topology, the map $\kappa^1 : \mathcal{T}_\pi^1 \rightarrow \mathcal{S}_\pi^1$ is isomorphic along $\mathcal{S}_\pi^1 - \mathcal{S}_\pi^0$ to the product*

$$\Pi \times \text{id} : \Gamma' \times \bar{J}_{C'}^n \longrightarrow \Gamma \times \bar{J}_C^n.$$

More precisely, there exists a smooth map $\psi : U \rightarrow \mathcal{S}_\pi^1$ whose image contains $\mathcal{S}_\pi^1 - \mathcal{S}_\pi^0$, there exists a smooth map $\psi' : U' \rightarrow \bar{J}_C^n$, and there exists a Cartesian diagram

$$\begin{array}{ccc} \mathcal{T}_\pi^1 \times_{\mathcal{S}_\pi^1} U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \Gamma' \times U' & \xrightarrow{\Pi \times \text{id}} & \Gamma \times U' \end{array}$$

in which the top morphism is the projection and the two vertical maps are open embeddings. Moreover, the Cartesian diagram induced by that diagram via the closed embedding $\gamma \hookrightarrow \Gamma$ is

$$(3.10) \quad \begin{array}{ccc} (\mathcal{T}_\pi^1 - \mathcal{T}_\pi^0) \times_{\mathcal{S}_\pi^1} U & \longrightarrow & (\mathcal{S}_\pi^1 - \mathcal{S}_\pi^0) \times_{\mathcal{S}_\pi^1} U \\ \downarrow & & \downarrow \\ \gamma' \times U' & \longrightarrow & \gamma \times U'. \end{array}$$

The two vertical morphisms in the latter diagram are isomorphisms.

PROOF. We first prove the result supposing that $-n \geq 3p + \delta - 1$, where p is the arithmetic genus of C . Set once and for all $m = 1 - p - n - \delta$, so that $m \geq 2p$. Let U be the open subset of $\Gamma \times \mathbf{H}_Q^m$ parametrizing pairs of subschemes of C with disjoint supports. There is

a natural map $U \rightarrow \mathbf{H}^{m+\delta}$, which is étale by the infinitesimal criterion. Since C is locally planar, hence Gorenstein, and $m + \delta \geq 2p - 1$, the Abel map $\mathcal{A} : \mathbf{H}^{m+\delta} \rightarrow \bar{J}_C^n$ is smooth. The image of the composition

$$U \rightarrow \mathbf{H}^{m+\delta} \rightarrow \bar{J}_C^n$$

is clearly contained in S_π^1 . Let $\psi : U \rightarrow S_\pi^1$ denote this composition.

We wish to show that the image of ψ contains $S_\pi^1 - S_\pi^0$. In analogy with the usual terminology, say a point P is a *base point* of a torsion-free, rank 1 sheaf \mathcal{I} on C if the support of every subscheme Y in the fiber $\mathcal{A}^{-1}([I])$ of the Abel map contains P . Let V be the open subset of $\mathbf{H}_Q^{\delta-1} \times \mathbf{H}_Q^m$ consisting of pairs of subschemes with disjoint supports. We obtain a canonical morphism $\phi : V \rightarrow S_\pi^0$ by composing the canonical étale morphism $V \rightarrow \mathbf{H}_Q^{m+\delta-1}$ with the restriction $\mathbf{H}_Q^{m+\delta-1} \rightarrow \bar{J}_C^{n+1}$ of the Abel map, which factors through S_π^0 . For any point P in the support of Q , form the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\otimes \mathcal{M}_P} & U \\ \phi \downarrow & & \downarrow \psi \\ S_\pi^{0,n+1} & \xrightarrow{\otimes \mathcal{M}_P} & \bar{J}_C^n, \end{array}$$

where \mathcal{M}_P is the ideal sheaf of the point P . By definition, $S_\pi^1 - S_\pi^0$ is the union of the images of the bottom maps for all P in the support of Q . It thus remains to show that ϕ is surjective. To prove this, it is sufficient to show that the map ϕ restricted to $V \cap \{(\delta - 1)a\} \times \mathbf{H}_Q^m$ is surjective, where a is a point of C contained in the regular locus. Since $m + \delta - 1 \geq 2p - 1$, for any torsion-free, rank 1 sheaf \mathcal{I} on C of degree m , the group $H^0(C, \mathcal{I}((\delta - 1)a))$ does not vanish. Furthermore, since $m \geq 2p$, any such \mathcal{I} does not have base points, so we can choose a section in $H^0(C, \mathcal{I}((\delta - 1)a))$ which does not vanish at a . This implies that the restriction of ϕ is surjective, hence we are done.

Let $U' = \mathbf{H}_Q^m$, and let $\psi' : U' \rightarrow \bar{J}_C^n$ be the restriction of the Abel map. From the canonical isomorphism $\mathbf{H}_Q^m \xrightarrow{\sim} \mathbf{H}_Q^m$, we get the desired embedding $U \hookrightarrow \Gamma \times U'$. It remains to find the asserted open embedding of $\mathcal{T}_\pi^1 \times_{S_\pi^1} U$ in $\Gamma' \times U'$. Let U'' be the open subset of $\Gamma' \times U'$ parametrizing pairs of subschemes with disjoint supports. It will turn out that U'' is the image of the embedding in question.

We first define a map $U'' \rightarrow \mathcal{T}_\pi^1 \times_{S_\pi^1} U$. To start with, the map $\Gamma' \times U' \rightarrow \Gamma \times U'$ obviously restricts to a map $U'' \rightarrow U$. Now consider a T -point $T \rightarrow U''$ of U'' . Let Y be the subscheme of C_T of length $m + \delta$ corresponding to the composition

$$T \rightarrow U'' \rightarrow U \rightarrow \mathbf{H}^{m+\delta}$$

and let Y' be the subscheme of length $m + \delta$ of C'_T corresponding to

$$T \rightarrow U'' \rightarrow \mathbf{H}'^{m+\delta}.$$

Let Y and Y' have ideal sheaves \mathcal{I} and \mathcal{I}' respectively. We can form the following diagram:

$$(3.11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \xrightarrow{h} & \pi_* \mathcal{I}' & \longrightarrow & \mathcal{B}_T \longrightarrow 0 \\ & & \downarrow i & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_{C_T} & \longrightarrow & \pi_* \mathcal{O}_{C'_T} & \longrightarrow & \mathcal{B}_T \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_Y & \longrightarrow & \pi_* \mathcal{O}_{Y'} & & \end{array}$$

We prove that the map $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_{Y'}$ is an isomorphism. By Nakayama's lemma and since $\pi_* \mathcal{O}_{Y'}$ is flat over T , it is sufficient to prove this on the fibers, so assume for a moment that $T = \text{Spec } k$. Then $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_{Y'}$ is clearly an isomorphism off Q . If for some i , no point P_i^j is in the support of Y' , then P_i is not in the support of Y either, so both \mathcal{O}_{Y, P_i} and $(\pi_* \mathcal{O}_{Y'})_{P_i}$ are zero. Otherwise, both \mathcal{O}_Y and $(\pi_* \mathcal{O}_{Y'})_{P_i}$ have length one over k , so that $\mathcal{O}_{Y, P_i} \rightarrow (\pi_* \mathcal{O}_{Y'})_{P_i}$ is again an isomorphism. A simple diagram chase now shows that the map $\pi_* \mathcal{I}' \rightarrow \mathcal{B}_T$ is surjective. Hence the first row of diagram (3.11) is a presentation, which lies in \mathcal{T}_π^1 since \mathcal{I} lies in \mathcal{S}_π^1 . Hence we get a map $U'' \rightarrow \mathcal{T}_\pi^1$, which we combine with $U'' \rightarrow U$ to get a map $U'' \rightarrow \mathcal{T}_\pi^1 \times_{\mathcal{S}_\pi^1} U$.

We now show that this map $U'' \rightarrow \mathcal{T}_\pi^1 \times_{\mathcal{S}_\pi^1} U$ is a monomorphism. Suppose the given T -point of U'' corresponds to the pair (Y'_1, Y'_2) of subschemes of C'_T with disjoint supports. The image of this T -point in U is a pair of subschemes (Y_1, Y_2) of C_T with disjoint supports; let Y be their sum, corresponding to an ideal sheaf $i : \mathcal{I} \hookrightarrow \mathcal{O}_{C_T}$. Since Y_2 is supported on $C_T - Q_T$, we can recover Y'_2 from Y_2 . Furthermore, the image of the given T -point of U'' in \mathcal{T}_π^1 corresponds to a presentation $h : \mathcal{I} \rightarrow \pi_* \mathcal{I}'$ having the same source as i since by assumption both i and h come from a T -point of U'' . Since diagram (3.11) is a pushout diagram, it is uniquely determined by i and h . Hence we can recover $\pi_* \mathcal{I}' \rightarrow \pi_* \mathcal{O}_{C'_T}$, which corresponds an embedding $\mathcal{I}' \rightarrow \mathcal{O}_{C'_T}$ on C'_T of colength $m + \delta$, that is, to a subscheme Y' on C'_T of length $m + \delta$. By definition of the map $U'' \rightarrow \mathcal{T}_\pi^1 \times_{\mathcal{S}_\pi^1} U$, the subscheme Y' is the sum of Y'_1 and Y'_2 . Since we have already recovered Y'_2 , we see that Y'_1 is uniquely determined, so that $U'' \rightarrow \mathcal{T}_\pi^1 \times_{\mathcal{S}_\pi^1} U$ is indeed a monomorphism.

We must show that we can lift a T -point of $\mathcal{T}_\pi^1 \times_{\mathcal{S}_\pi^1} U$ to a T -point of U'' . By descent theory, it is sufficient to do this up to an étale morphism. Let the given T -point of \mathcal{T}_π^1 be representable by a presentation $h : \mathcal{I} \rightarrow \pi_* \mathcal{I}'$. According to Lemma 3.7, and since it is flat

over T , the cokernel of h is invertible on Q_T . Since Q_T is finite and flat over T , we may assume, after an étale base change, if necessary, that the cokernel of h is \mathcal{B}_T . Let the given T -point of U correspond to a pair of subschemes of C_T with disjoint supports whose sum corresponds to an embedding $i : \mathcal{I}'' \rightarrow \mathcal{O}_{C_T}$ of colength $m + \delta$. Once again, since \mathcal{I}'' and \mathcal{I} must have the same image in \mathcal{S}_π , we may assume that $\mathcal{I}'' = \mathcal{I}$, after an étale base change, if necessary.

Form the following pushout diagram:

$$(3.12) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \xrightarrow{h} & \pi_* \mathcal{I}' & \longrightarrow & \mathcal{B}_T \longrightarrow 0 \\ & & \downarrow i & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_{C_T} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{B}_T \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{F} & = & \mathcal{F} & & \end{array}$$

Suppose that $\mathcal{E} = \pi_* \mathcal{I}''$ for some rank 1, torsion-free sheaf \mathcal{I}'' on C'_T . Then the embedding $\mathcal{O}_{C_T} \rightarrow \pi_* \mathcal{I}''$ is a presentation. However, there is a unique presentation with source \mathcal{O}_{C_T} , namely $\mathcal{O}_{C_T} \rightarrow \pi_* \mathcal{O}_{C'_T}$, so that we must have $\mathcal{E} = \pi_* \mathcal{O}_{C'_T}$. Then the second vertical sequence defines an embedding $\mathcal{I}' \hookrightarrow \mathcal{O}_{C'_T}$ which, along with the original T -point of U , defines a T -point of U'' which lifts the given T -point of $\mathcal{T}_\pi^1 \times_{\mathcal{S}_\pi^1} U$, so we are done. It remains to show that $\mathcal{E} = \pi_* \mathcal{I}''$ for some rank 1, torsion-free sheaf \mathcal{I}'' on C'_T . We first prove this on the fibers; it will then be sufficient to see that \mathcal{E} is a $\pi_* \mathcal{O}_{C'_T}$ -module. So assume for the moment that $T = \operatorname{Spec} k$. The assertion is local on C , and trivially holds off Q . If $\operatorname{Supp} \mathcal{F}$ does not contain P , then in diagram (3.12), the lower sequence is equal to the upper one at P , so the assertion holds there too. Otherwise, let $R = \mathcal{O}_{C,P}$ and $R' = (\pi_* \mathcal{O}_{C'})_P$. Let $I' = (\pi_* \mathcal{I}')_P$, and note that $\mathcal{I}_P \simeq \mathfrak{m}_P$. Let $J = \mathfrak{m}_P R$. Then $J = (\pi_* \mathcal{J}')_P$, where \mathcal{J}' is the ideal sheaf on C' of $\pi^{-1}(P)$. Hence the quotient $A = R'/J$ is an Artin ring of dimension 2 over k . By (2.5), the presentation h factors through $\pi_*(\mathcal{I}^\pi) \rightarrow \pi_* \mathcal{I}'$, where \mathcal{I}^π is the torsion-free, rank 1 sheaf on C' generated by \mathcal{I} on C . The cokernel of $\pi_*(\mathcal{I}^\pi) \rightarrow \pi_* \mathcal{I}'$ must have length one. Indeed, by Proposition 3.4 the sheaf \mathcal{I} must be equal to $(\pi_P)_* \mathcal{I}^P$ for some invertible sheaf \mathcal{I}^P on the blow-up C_P of C at P ; then h factors through $(\pi_P)_* \mathcal{I}^P \rightarrow (\pi_P)_*(\pi')^* \mathcal{I}^P$, where $\pi' : C' \rightarrow C_P$, and the latter embedding has colength $\delta - 1$. Hence, localizing at P and noting that $[\pi_*(\mathcal{I}^\pi)]_P$ is equal to J , we get an embedding $J \subseteq I'$ whose cokernel N is an A -module of length one over k . There is thus up to scalars a unique embedding $N \hookrightarrow A$.

We will construct an embedding $\alpha : I' \rightarrow R'$ making the following diagram commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & I' & \longrightarrow & N \longrightarrow 0 \\ & & \parallel & & \downarrow \alpha & & \downarrow \\ 0 & \longrightarrow & J & \longrightarrow & R' & \longrightarrow & A \longrightarrow 0. \end{array}$$

Since $C' \rightarrow C$ factors through the blow-up C_P of C at P , the sheaf \mathcal{J}' is invertible on C' . Hence we may pick a generator e of J as an R' -module. Let $g \in I'$ be such that its image in N generates N . Then eg lies in J and is non-zero since I' is torsion-free. Hence we may identify eg with an element b of R' lying in J . Since e generates J as an R' -module, there exists $g' \in R'$ such that $b = eg'$. Furthermore, g' is uniquely determined since $\text{Ann}_{R'}(e) = 0$ because J is torsion-free. Sending g to g' and sending any element of $J \subseteq I'$ to itself in $J \subseteq R'$ defines an R' -module map $I' \rightarrow R'$ making the above diagram commute, and it is easy to verify that this map is independent of the choice of generators and is injective. The colength of I' in R' must be one, hence I' must be equal to the maximal ideal $\mathfrak{m}_{P'}$ of some points P' on C' mapping to P . Hence diagram (3.12) localizes at P to the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{m}_P & \longrightarrow & (\pi_* \mathfrak{m}_{P'})_P & \longrightarrow & \mathcal{B}_P \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_{C,P} & \longrightarrow & \mathcal{E}_P & \longrightarrow & \mathcal{B}_P \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{F}_P & \xlongequal{\quad} & \mathcal{F}_P & & \end{array}$$

In that diagram, $(\pi_* \mathcal{O}_{C'})_P$ fits in place of \mathcal{E}_P . However, a pushout diagram is unique, hence \mathcal{E}_P is isomorphic to $(\pi_* \mathcal{O}_{C'})_P$. Hence we proved that the fibers of \mathcal{E} are rank 1, torsion-free $(\pi_* \mathcal{O}_{C'})_P$ -modules. To show that \mathcal{E} is a $\pi_* \mathcal{O}_{C'_T}$ -module, consider the canonical map $\mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_{C_T}} \mathcal{K}_T$, where \mathcal{K} is the total quotient ring sheaf of \mathcal{O}_{C_T} . This map is injective on the fibers, since \mathcal{E} is injective on the fibers, and so is injective, since both \mathcal{E} and \mathcal{K} are flat over T . However, tensoring the second row of diagram (3.12) with \mathcal{K}_T , which is flat on C_T , we obtain that $\mathcal{E} \otimes_{\mathcal{O}_{C_T}} \mathcal{K}_T = \mathcal{K}_T$ since \mathcal{B}_T consists of torsion. Hence \mathcal{E} embeds in \mathcal{K}_T , which is a $\pi_* \mathcal{O}_{C'_T}$ -module, since π is birational. It is clear that \mathcal{F} is a $\pi_* \mathcal{O}_{C'_T}$ -module because $\Gamma' \rightarrow \Gamma$ is surjective on T -points. The quotient $\mathcal{K}_T / \pi_* \mathcal{I}'$ is also a $\pi_* \mathcal{O}_{C'_T}$ -module, and so is the image of \mathcal{E} in this quotient, because it's equal to \mathcal{F} . Hence \mathcal{E} itself is a $\pi_* \mathcal{O}_{C'_T}$ -module.

Finally, it is obvious that the two diagrams in the statement of the proposition are Cartesian. The morphism $(\mathcal{S}_\pi^1 - \mathcal{S}_\pi^0) \times_{\mathcal{S}_\pi^1} U \rightarrow \gamma \times U'$ is an isomorphism since $\gamma \times U'$ is contained in the image of U in $\Gamma \times U'$. Having proved the statement of the proposition for $-n$ large enough, note that, for any integer a , twisting by $\mathcal{O}_C(a)$ defines an isomorphism $\bar{J}_C^n \xrightarrow{\sim} \bar{J}_C^{n+b}$ where $b = \deg \mathcal{O}_C(a)$. Similarly, twisting by $\pi^* \mathcal{O}_C(a)$ defines a compatible

isomorphism $\bar{J}_{C'}^n \xrightarrow{\sim} \bar{J}_{C'}^{n+b}$ because $b = \deg \pi^* \mathcal{O}_C(a)$. Using this twisting as appropriate, we see that the result holds for any n . \square

Corollary 3.13. *Let C be a curve with double points and $\pi : C' \rightarrow C$ the blow-up of C at a point P . The morphism $\kappa^1 : \mathcal{T}_\pi^1 \rightarrow \mathcal{S}_\pi^1$ is a finite surjective birational morphism of integral varieties, and is isomorphic to the blow-up of \mathcal{S}_π^1 along $\mathcal{S}_\pi^1 - \mathcal{S}_\pi^0$. Moreover, the sheaf $\mathcal{B} = \kappa_*^1 \mathcal{O}_{\mathcal{T}_\pi^1} / \mathcal{O}_{\mathcal{S}_\pi^1}$ is invertible on $\Sigma = \text{Supp}(\mathcal{B})$.*

PROOF. Since C is locally planar, \bar{J}_C^n is integral. Hence so is \mathcal{S}_π^1 . Since C' is also locally planar, $\bar{J}_{C'}^{n+1}$ is integral. Since Pres_π^n is a \mathbb{P}^1 -bundle over $\bar{J}_{C'}^{n+1}$, according to Proposition 3.14, it is also integral. Hence so is \mathcal{T}_π^1 . Since κ is an isomorphism over \mathcal{S}_π^0 , it is birational and surjective. Hence so is κ^1 . The assertions that κ^1 is finite and that \mathcal{B} is invertible on Σ follow from the above Proposition and classical descent theory, as does the fact that κ^1 is isomorphic to the blow-up of \mathcal{S}_π^1 along $\mathcal{S}_\pi^1 - \mathcal{S}_\pi^0$. \square

Whereas the description of κ above is partial, the morphism κ' is satisfyingly described in the following proposition. Compare with Proposition 2.6.

Proposition 3.14. *Let C be an integral curve with double points and $\pi : C' \rightarrow C$ a partial normalization. Then the sheaf $\mathcal{F} = (\pi_* \mathbf{I}')|_{Q \times \bar{J}_{C'}^{n+1}}$ is locally free of rank 2 on $Q \times \bar{J}_{C'}^{n+1}$. Hence, κ' makes Pres_π^n a bundle over $\bar{J}_{C'}^{n+1}$ with fiber*

$$F_\pi = \text{Quot}_{H^0(Q', \mathcal{O}_{Q'})/Q/k}^\delta.$$

In particular, if $\delta = 1$, then $\text{Pres}_\pi^n = \mathbb{P}(\mathcal{F})$ is a \mathbb{P}_k^1 -bundle over $\bar{J}_{C'}^{n+1}$.

PROOF. It is sufficient to show \mathcal{F} is locally free of rank 2 on the reduced subscheme $Q_{\text{red}} \times \bar{J}_{C'}^{n+1}$. If \mathcal{I}' is a torsion-free, rank 1 sheaf on C' and P a point in Q_{red} , then the fiber $\mathcal{F}(P, [\mathcal{I}'])$ of \mathcal{F} at the point $(P, [\mathcal{I}'])$ of $Q_{\text{red}} \times \bar{J}_{C'}^{n+1}$ is equal to $(\pi_* \mathcal{I}')(P)$. Since $\pi_* \mathcal{I}'$ is not invertible at P , $\dim_k(\pi_* \mathcal{I}')(Q) \geq 2$, but since P is a double point of C , we must have $\dim_k(\pi_* \mathcal{I}')(P) = 2$, by Proposition 3.1(2). Since C' is locally planar, $\bar{J}_{C'}^{n+1}$ is reduced, hence \mathcal{F} is locally free of rank 2 on $\bar{J}_{C'}^{n+1}$. The rest of the proposition follows as in the proof of Proposition 2.6. \square

If $\pi : C' \rightarrow C$ is the blow-up of C at a double point P , we will need a more detailed description of $\text{Pres}_{\pi, C'}^n$ than that provided by Proposition 3.9. If Q' lies in the regular locus of C' , then $\text{Pres}_{\pi, C'}^n$ is adequately described in diagram (2.8). Otherwise, Q' is supported at a double point P' . Let $\pi'' : C'' \rightarrow C'$ be the blow-up of C' at P' . Then the somewhat

subtler situation is described in the following proposition.

Proposition 3.15. *Let the singularities of C be double points and let $\pi : C' \rightarrow C$ be the blow-up of C at P . Suppose Q' is supported at a double point P' of C' . Then*

$$\text{Pres}_{\pi, C'}^n = E \cup E',$$

where E and E' are irreducible.

The reduced subscheme E_{red} of E is equal to $(\kappa')^{-1}(\bar{J}_{C''}^{n+1})$. Also, E_{red} is canonically isomorphic to $\text{Pres}_{\pi''}^n$. The restriction $\kappa : E_{\text{red}} \rightarrow \bar{J}_{C'}^n$ is isomorphic to the canonical map $\text{Pres}_{\pi''}^n \rightarrow \bar{J}_{C'}^n$, under this identification. The morphism $(\kappa, \sigma) : E \rightarrow \bar{J}_{C'}^n \times Q'$ restricts on \mathcal{T}_{π}^1 to an isomorphism

$$E \cap \mathcal{T}_{\pi}^1 \xrightarrow{\sim} J_{C'}^n \times Q',$$

which is compatible with the canonical embedding of $J_{C'}^n$ inside $\text{Pres}_{\pi''}^n$.

Furthermore, κ' maps E' onto $\bar{J}_{C'}^{n+1}$ and κ maps E' into $\bar{J}_{C'}^n \subseteq \bar{J}_{C'}^n$. In particular, $E' \cap \mathcal{T}_{\pi}^1$ is empty.

PROOF. Let $\pi' : C'' \rightarrow C$ be the composition $\pi \circ \pi''$. Define E_r to be $(\kappa')^{-1}(\bar{J}_{C''}^{n+1})$. Then κ maps E_r into $\bar{J}_{C'}^n$. Indeed, the source of a presentation $h : \mathcal{I} \rightarrow \pi'_* \mathcal{I}''$ cannot be invertible at P , since then we would have $\pi'_* \mathcal{I}'' = \pi_*(\pi''_* \mathcal{I}'') = \pi_* \pi^* \mathcal{I}$, which is impossible. Since \mathcal{I} is not invertible at P , it lies in $\bar{J}_{C'}^n$, by Proposition 3.4. Hence $E_r \subseteq \text{Pres}_{\pi, C'}^n$.

Consider the partial section $s : J_{C'}^{n+1} \rightarrow \text{Pres}_{\pi}^n$ of κ' defined by sending \mathcal{L}' to the presentation $h : \pi_*(\mathcal{M}_{P'} \mathcal{L}') \rightarrow \pi_* \mathcal{L}'$, where $\mathcal{M}_{P'}$ is the maximal ideal sheaf of P' on C' . Let E'_r be the closure of the image of this section. Then κ maps E'_r to $\bar{J}_{C'}^n \subseteq \bar{J}_{C'}^n$, since the source $\pi_*(\mathcal{M}_{P'} \mathcal{I})$ of h is not invertible on C at P and $\mathcal{M}_{P'} \mathcal{I}$ is not invertible on C' at P' . Hence $E'_r \subseteq \text{Pres}_{\pi, C'}^n$.

Let $h : \pi_* \mathcal{I}' \rightarrow \pi_* \mathcal{L}'$ be any presentation on C lying in $\text{Pres}_{\pi, C'}^n$. If \mathcal{L}' is not invertible on C' at P' , then h lies in E_r . On the other hand, if \mathcal{L}' is invertible at P' , then we must have that $\mathcal{I}' = \mathcal{M}_P \mathcal{L}'$. Hence in this case, h lies in E'_r . Thus we have shown that the reduced subscheme underlying $\text{Pres}_{\pi, C'}^n$ is equal to $E_r \cup E'_r$. Let E and E' be the corresponding components of $\text{Pres}_{\pi, C'}^n$. It is clear that $E_{\text{red}} = E_r$ and $E'_{\text{red}} = E'_r$.

Let $\text{Pres}_{\pi''}^n \rightarrow \text{Pres}_{\pi}^n$ be the morphism sending a presentation h'' on C' to $\pi''_* h''$ on C . It is then clear that the image of this morphism is precisely E_{red} and that the composition $\text{Pres}_{\pi''}^n \rightarrow E_{\text{red}} \subseteq \text{Pres}_{\pi, C'}^n \rightarrow \bar{J}_{C'}^n$ is equal to the canonical map.

Finally, consider $(\kappa, \sigma) : E \rightarrow \bar{J}_{C'}^n \times Q'$. We construct an inverse of this morphism over $\bar{J}_{C'}^n \times Q'$ as follows. If \mathcal{L}' is an invertible sheaf on $C' \times T$ and $q' : T \rightarrow Q'$ a T -point of Q' , then form the canonical map $h' : \mathcal{L}' \rightarrow \mathcal{M}_{q'}^{\vee} \mathcal{L}'$, where $\mathcal{M}_{q'}$ denotes the ideal sheaf of the image of the section $(q', \text{id}) : T \rightarrow C' \times T$ and $\mathcal{M}_{q'}^{\vee} = \mathcal{H}om_{\mathcal{O}_{C'_T}}(\mathcal{M}_{q'}, \mathcal{O}_{C'_T})$. Then the cokernel of h' is included in Q'_T and is flat, of length one over T , so that $\pi_* h'$ is a presentation, which

lies in $\text{Pres}_{\pi, C'}^n \cap \mathcal{T}_\pi^1$. The composition $(\kappa, \sigma) \circ \alpha$ is clearly the identity. As for the opposite composition, note that if we have a presentation $h : \pi_* \mathcal{L}' \rightarrow \pi_* \mathcal{I}'$ with \mathcal{L}' invertible on C' , then we can tensor by \mathcal{L}'^{-1} on C'_T to obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{C'_T} \longrightarrow \mathcal{I}' \otimes \mathcal{L}'^{-1} \longrightarrow \mathcal{N} \longrightarrow 0,$$

where \mathcal{N} is flat and finite of length one over T , and supported on Q'_T . It is easy to see that $\mathcal{I}' \otimes \mathcal{L}'^{-1}$ must then be equal to $\mathcal{M}_{q'}^\vee$ for some T -point of Q' . Further, the image of $J_{C'}^n \times Q' \rightarrow E$ is equal to $E \cap \mathcal{T}_\pi^1$ \square

We should note that the above proposition shows that the morphism $\kappa : \text{Pres}_\pi^n \rightarrow \bar{J}_C^n$ is not finite if C' is *not* regular at Q' . Indeed, say C has arithmetic genus p . Then E'_{red} has dimension $p-1$ and κ maps it into $\bar{J}_{C''}^n$, which has dimension $p-2$. However, the restriction κ^1 of κ to \mathcal{T}_π^1 is of course finite according to Corollary 3.13.

4. AUTODUALITY

Let C be an S -curve, and consider the Abel map in degree 1:

$$\mathcal{A}^1 : C = \text{Hilb}_{C/S}^1 \longrightarrow \bar{J}_C^{-p}.$$

Let Φ denote the induced map

$$\Phi : \text{Pic}_{\bar{J}_C^{-p}/S}^0 \longrightarrow J_C^{1-p} = \text{Pic}_{C/S}^0$$

given by pullback of invertible sheaves. If C is smooth over S , classical autoduality asserts that Φ is an isomorphism; see for example [18, VI, Theorem 3]. In what follows, we generalize this result to curves with double points. First, define what will be the inverse Ψ of Φ as follows. Recall from [17] the definition of the determinant of cohomology $\mathcal{D}_f(\mathcal{F})$ associated to a morphism $f : X \rightarrow Y$ and a quasi-coherent sheaf \mathcal{F} on X . Let \mathbf{I} be a universal sheaf on $C \times_S \bar{J}_C^{-p}$ and let \mathbf{L} be a universal sheaf on $C \times_S J_C^{1-p}$. Let p_{12}, p_{13} and p_{23} denote the projections from $C \times_S \bar{J}_C^{-p} \times_S J_C^{1-p}$ to $C \times_S \bar{J}_C^{-p}$, $C \times_S J_C^{1-p}$ and $\bar{J}_C^{-p} \times_S J_C^{1-p}$ respectively. To simplify notation, we simply write \mathcal{D} for the determinant of cohomology $\mathcal{D}_{p_{23}}$ associated to the projection p_{23} . Let the sheaf Υ on $\bar{J}_C^{-p} \times_S J_C^{1-p}$ be given by

$$\Upsilon = \mathcal{D}(p_{12}^* \mathbf{I} \otimes p_{13}^* \mathbf{L})^{-1} \otimes \mathcal{D}(p_{13}^* \mathbf{L}) \otimes \mathcal{D}(p_{12}^* \mathbf{I}) \otimes \mathcal{D}(\mathcal{O}_{C \times_S \bar{J}_C^{-p} \times_S J_C^{1-p}})^{-1}.$$

Then the morphism

$$\Psi : J_C^{1-p} \longrightarrow \text{Pic}_{\bar{J}_C^{-p}}^0$$

is defined on T -points by sending $t : T \rightarrow J_C^{1-p}$ to the T -point of $\text{Pic}_{\bar{J}_C^{-p}}^0$ corresponding to

the line bundle on $\bar{J}_C^{-p} \times T$ given by the pullback of Υ by the map

$$\bar{J}_C^{-p} \times_S T \xrightarrow{(\text{id}, t)} \bar{J}_C^{-p} \times_S J_C^{1-p}.$$

The map Ψ could *a priori* depend on the choice of universal sheaves \mathbf{I} and \mathbf{L} , but it will be shown in Theorem 4.6 that it is the inverse to Φ for curves with doubles points, hence in fact does not depend on this choice. The easy part of autoduality is the following lemma.

Lemma 4.1. *The composition $\Phi \circ \Psi$ is the identity on J_C^{1-p} .*

PROOF. Since all the maps involved are defined by their actions on T -points, it is natural to look at functors. Let $t : T \rightarrow J_C^{1-p}$ be a T -point of J_C^{1-p} corresponding to a line bundle \mathcal{M} on $C \times_S T$, that is, so that $t^*\mathbf{L} = \mathcal{M}$. To simplify notation, we will always denote by p_{ij} the projections from a triple product $X_1 \times_S X_2 \times_S X_3$ obtained by base-changing $p_{23} : C \times_S \bar{J}_C^{-p} \times_S J_C^{1-p} \rightarrow \bar{J}_C^{-p} \times_S J_C^{1-p}$. Since the determinant of cohomology commutes with base change, we have

$$\Psi([\mathcal{M}]) = \mathcal{D}(p_{12}^*\mathbf{I} \otimes p_{13}^*\mathcal{M})^{-1} \otimes \mathcal{D}(p_{13}^*\mathcal{M}) \otimes \mathcal{D}(p_{12}^*\mathbf{I}) \otimes \mathcal{D}(\mathcal{O}_{C \times_S \bar{J}_C^{-p} \times_S T})^{-1},$$

where \mathcal{D} is taken with respect to the morphism $p_{23} : C \times_S \bar{J}_C^{-p} \times_S T \rightarrow \bar{J}_C^{-p} \times_S T$. The pullback of \mathbf{I} through the Abel map $C \times_S C \rightarrow C \times_S \bar{J}_C^{-p}$ is equal to the ideal sheaf Δ of the diagonal. Hence, we have

$$\Phi(\Psi([\mathcal{M}])) = \mathcal{D}(\Delta_T \otimes p_{13}^*\mathcal{M})^{-1} \otimes \mathcal{D}(p_{13}^*\mathcal{M}) \otimes \mathcal{D}(\Delta_T) \otimes \mathcal{D}(\mathcal{O}_{C \times_S C \times_S T})^{-1},$$

where \mathcal{D} is taken with respect to the morphism $p_{23} : C \times_S C \times_S T \rightarrow C \times_S T$. Applying the determinant of cohomology to the sequence

$$0 \longrightarrow \Delta_T \longrightarrow \mathcal{O}_{C \times_S C \times_S T} \longrightarrow \mathcal{O}_{\Delta_T} \longrightarrow 0,$$

we obtain

$$\mathcal{D}(\Delta_T) = \mathcal{D}(\mathcal{O}_{C \times_S C \times_S T}).$$

Furthermore, tensoring the above sequence with $p_{23}^*\mathcal{M}$ and taking the determinant of cohomology, we obtain

$$\mathcal{D}(\Delta_T \otimes p_{13}^*\mathcal{M})^{-1} = \mathcal{M} \otimes \mathcal{D}(p_{13}^*\mathcal{M})^{-1}.$$

Hence we get $\Phi(\Psi([\mathcal{M}])) = [\mathcal{M}]$, as needed to be shown. \square

The proof of autoduality for double points we present further below proceeds by induction, using partial normalizations C' of C and the presentation functor as a bridge between the compactified Jacobians of C and C' , as codified in the following lemma. Let the morphisms

$\Phi' : \text{Pic}_{\bar{J}_{C'}^{-p}/S}^0 \rightarrow J_{C'}^{\delta+1-p}$ and $\Psi' : J_{C'}^{\delta+1-p} \rightarrow \text{Pic}_{\bar{J}_{C'}^{-p}/S}^0$ be defined for C' as they are for C .

Lemma 4.2. *Let $\pi : C' \rightarrow C$ be a partial normalization of C , and assume that Q has fibers of length one over S . Then the following diagram commutes:*

$$(4.3) \quad \begin{array}{ccc} J_C^{1-p} & \xrightarrow{\pi^*} & J_{C'}^{\delta+1-p} \\ \downarrow \Psi & & \downarrow \Psi' \\ & & \text{Pic}_{\bar{J}_{C'}^{-p}}^0 \\ \downarrow & & \downarrow (\kappa')^* \\ \text{Pic}_{\bar{J}_C^{-p}}^0 & \xrightarrow{\kappa^*} & \text{Pic}_{\text{Pres}_\pi^{-p}}^0 \end{array}$$

PROOF. We prove this once again on the level of functors, using the same notational conventions as in the proof of the previous lemma. Let $t : T \rightarrow J_C^{1-p}$ be a T -point of J_C^{1-p} corresponding to a line bundle \mathcal{M} on $C \times_S T$. Then $\Psi \circ \kappa^*([\mathcal{M}])$ is defined by the line bundle

$$(4.4) \quad \mathcal{D}(p_{12}^* \kappa^* \mathbf{I} \otimes p_{13}^* \mathcal{M})^{-1} \otimes \mathcal{D}(p_{13}^* \mathcal{M}) \otimes \mathcal{D}(\kappa^* \mathbf{I}) \otimes \mathcal{D}(\mathcal{O}_{C \times_S \text{Pres}_\pi^{-p} \times_S T})^{-1},$$

where \mathcal{D} is taken with respect to the morphism $p_{23} : C \times_S \text{Pres}_\pi^{-p} \times_S T \rightarrow \text{Pres}_\pi^{-p} \times_S T$.

On the other hand, $\kappa'^* \circ \Psi' \circ \pi^*([\mathcal{M}])$ is defined by the line bundle

$$\mathcal{D}'(p_{12}^* \kappa'^* \mathbf{I}' \otimes p_{13}^* \pi^* \mathcal{M})^{-1} \otimes \mathcal{D}(p_{13}^* \pi^* \mathcal{M}) \otimes \mathcal{D}(p_{12}^* \kappa'^* \mathbf{I}') \otimes \mathcal{D}(\mathcal{O}_{C' \times_S \text{Pres}_\pi^{-p} \times_S T})^{-1}$$

on $\text{Pres}_\pi^{-p} \times_S T$, where the determinant of cohomology is taken with respect to the projection $p_{23} : C' \times_S \text{Pres}_\pi^{-p} \times_S T \rightarrow \text{Pres}_\pi^{-p} \times_S T$ and \mathbf{I}' is a universal torsion-free, rank 1 sheaf on $C' \times_S \bar{J}_{C'}^{-p}$. Because π is finite, the above line bundle is equal to

$$(4.5) \quad \mathcal{D}(p_{12}^* \pi_* \kappa'^* \mathbf{I}' \otimes p_{13}^* \mathcal{M})^{-1} \otimes \mathcal{D}(p_{13}^* \pi_* \pi^* \mathcal{M}) \otimes \mathcal{D}(r_{12}^* \pi_* \kappa'^* \mathbf{I}') \otimes \mathcal{D}(\pi_* \mathcal{O}_{C' \times_S \text{Pres}_\pi^{-p} \times_S T})^{-1},$$

where \mathcal{D} is taken with respect to the morphism $p_{23} : C \times_S \text{Pres}_\pi^{-p} \times_S T \rightarrow \text{Pres}_\pi^{-p} \times_S T$.

From the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \pi_* \mathcal{O}_{C'} \longrightarrow \mathcal{B} \longrightarrow 0,$$

we obtain

$$\mathcal{D}(\pi_* \mathcal{O}_{C'}) = \mathcal{D}(\mathcal{O}_C) \otimes \mathcal{D}(\mathcal{B}).$$

Tensoring the sequence by $p_{13}^* \mathcal{M}$, we obtain

$$\mathcal{D}(p_{13}^* \pi_* \pi^* \mathcal{M}) = \mathcal{D}(p_{13}^* \mathcal{M}) \otimes \mathcal{D}(\mathcal{B} \otimes p_{13}^* \mathcal{M}).$$

The sheaf \mathcal{B} on $C \times_S \text{Pres}_\pi^{-p} \times_S T$ is supported on $Q \times_S \text{Pres}_\pi^{-p} \times_S T$, which is isomorphic to $\text{Pres}_\pi^{-p} \times_S T$ by hypothesis. In fact, \mathcal{B} is locally free of rank δ on its support, where δ is the genus change of π . Hence we get

$$\mathcal{D}(\mathcal{B}) = \bigwedge^\delta \mathcal{B}$$

and

$$\mathcal{D}(\mathcal{B} \otimes p_{13}^* \mathcal{M}) = \bigwedge^\delta \mathcal{B} \otimes q^*(\mathcal{M}|_{Q_T}),$$

where $q : \text{Pres}_\pi^{-p} \times_S T \rightarrow T$ is the projection and where we identify Q_T with T . Consider now the universal presentation on $C \times_S \text{Pres}_\pi^{-p}$ and its cokernel:

$$0 \longrightarrow \kappa^* \mathbf{I} \longrightarrow \pi_* \kappa'^* \mathbf{I}' \longrightarrow \mathcal{N} \longrightarrow 0.$$

We thus have

$$\mathcal{D}(p_{12}^* \pi_* \kappa'^* \mathbf{I}') = \mathcal{D}(p_{12}^* \kappa^* \mathbf{I}) \otimes \mathcal{D}(p_{12}^* \mathcal{N}).$$

Further, tensoring the above sequence with $p_{13}^* \mathcal{M}$ and taking the determinant of cohomology, we obtain

$$\mathcal{D}(p_{12}^* \pi_* \kappa'^* \mathbf{I}' \otimes p_{13}^* \mathcal{M}) = \mathcal{D}(p_{12}^* \kappa^* \mathbf{I} \otimes p_{13}^* \mathcal{M}) \otimes \mathcal{D}(p_{13}^* \mathcal{M} \otimes p_{12}^* \mathcal{N}).$$

The sheaf \mathcal{N} on $C \times_S \text{Pres}_\pi^{-p} \times_S T$ is supported on $Q \times_S \text{Pres}_\pi^{-p} \times_S T$, and has fibers of Euler characteristic δ over T . Hence we get

$$\mathcal{D}(p_{12}^* \mathcal{N}) = \bigwedge^\delta \mathcal{N}$$

and

$$\mathcal{D}(p_{13}^* \mathcal{M} \otimes p_{12}^* \mathcal{N}) = \bigwedge^\delta \mathcal{N} \otimes q^*(\mathcal{M}|_{Q_T}).$$

Substituting the above equalities in (4.5), we find that $\kappa'^* \circ \Psi' \circ \pi^*([\mathcal{M}])$ is defined by the same line bundle as in (4.4). Hence we are done. \square

Assume for the rest of this section that S is the spectrum of an algebraically closed field. The proof of the following theorem compares the kernels of the maps π^* and κ^* to deduce that, under the stated conditions, Ψ is in fact an isomorphism.

Theorem 4.6. *Suppose the curve C has only double points. Then the maps Φ and Ψ are isomorphisms inverse to each other.*

PROOF. The proof proceeds by induction on the genus deficiency δ of C . If $\delta = 0$, then C is smooth and the assertion is simply classical autoduality; see [18, VI, Theorem 3]. Assume the result holds for all curves with double points mapping to C (but different from C). Let $\pi : C' \rightarrow C$ be a partial normalization with genus change one. Then C' has double points according to Lemma 3.3(1), so autoduality holds for C' . Form the commutative diagram (4.3). By Proposition 3.14, κ' is a \mathbb{P}^1 -bundle, hence $\kappa'^* : \text{Pic}_{\bar{J}_{C'}^{-p}}^0 \rightarrow \text{Pic}_{\text{Pres}_{\pi}^{-p}}^0$ is an isomorphism. Moreover, Ψ' is an isomorphism by the induction hypothesis. By Proposition 2.3, the map π^* is surjective in the *fppf* topology. Hence κ^* must also be surjective in the *fppf* topology. Let $G = G_{\pi}$ be the kernel of π^* , as in the exact sequence (2.4), and let the functor \underline{K}_{κ} denote the kernel of κ^* . Then the canonical morphism $G \rightarrow \underline{K}_{\kappa}$ is injective on T -points since Ψ has a one-sided inverse Φ according to Lemma 4.1.

Consider the restriction $\kappa^1 : \mathcal{T}_{\pi}^1 \rightarrow \mathcal{S}_{\pi}^1$ of κ and form the diagram

$$(4.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & G & \longrightarrow & \text{Pic}_C^0 & \longrightarrow & \text{Pic}_{C'}^0 \longrightarrow 0 \\ & & \downarrow & & \Psi \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{K}_{\kappa} & \longrightarrow & \text{Pic}_{\bar{J}_C^{-p}}^0 & \xrightarrow{\kappa^*} & \text{Pic}_{\text{Pres}_{\pi}^{-p}}^0 \longrightarrow 0 \\ & & \downarrow & & i^* \downarrow & & \downarrow j^* \\ 0 & \longrightarrow & \underline{K}^1 & \longrightarrow & \underline{\text{Pic}}_{\mathcal{S}_{\pi}^1}^0 & \xrightarrow{(\kappa^1)^*} & \underline{\text{Pic}}_{\mathcal{T}_{\pi}^1}^0 \end{array}$$

where the maps i^* and j^* are induced by the inclusions $i : \mathcal{S}_{\pi}^1 \hookrightarrow \bar{J}_C^{-p}$ and $j : \mathcal{T}_{\pi}^1 \hookrightarrow \text{Pres}_{\pi}^{-p}$. It suffices to prove that $\underline{K}_{\kappa} = G$. Indeed, the monomorphism $G \rightarrow \underline{K}_{\kappa}$ of group schemes will then be an isomorphism, so that by the 5-lemma, Ψ will also be an isomorphism, whose inverse must be Φ .

By [1, Theorem 9], \bar{J}_C^{-p} is Cohen-Macaulay, hence satisfies Serre's condition S_2 . According to (3.6), the codimension of the complement of \mathcal{S}_{π}^1 in \bar{J}_C is at least two, so by Lemma 1.17 we have $(\sigma_{\mathcal{S}_{\pi}^1})_* \mathcal{O}_{\mathcal{S}_{\pi}^1} = \mathcal{O}_S$. According to Corollary 3.13, the sheaf $\mathcal{B} = \kappa_*^1 \mathcal{O}_{\mathcal{T}_{\pi}^1} / \mathcal{O}_{\mathcal{S}_{\pi}^1}$ is invertible on $\Sigma = \text{Supp}(\mathcal{B}) \subseteq \mathcal{S}_{\pi}^1$, so we can apply Theorem 1.11 to κ^1 . We obtain that the kernel \underline{K}_{κ}^1 of $\underline{\text{Pic}}_{\mathcal{S}_{\pi}^1}^0 \rightarrow \underline{\text{Pic}}_{\mathcal{T}_{\pi}^1}^0$ is equal to $\underline{G}_{\bar{\kappa}^1}$, where $\bar{\kappa}^1$ is the restricted morphism $\Sigma' \rightarrow \Sigma$, with $\Sigma' = (\kappa^1)^{-1}(\Sigma) \subseteq \mathcal{T}_{\pi}^1$. Set-theoretically, Σ coincides with $\mathcal{S}_{\pi}^1 - \mathcal{S}_{\pi}^0$, since κ^1 is an isomorphism exactly on \mathcal{S}_{π}^0 . Applying Theorem 1.11 once more, and by flat descent, we must have $\Sigma = \mathcal{S}_{\pi}^1 - \mathcal{S}_{\pi}^0$ scheme-theoretically as well. But by formula (3.5), we have $\mathcal{S}_{\pi}^1 - \mathcal{S}_{\pi}^0 = \bar{J}_{C'}^{-p} \cap \mathcal{S}_{\pi}^1$. Hence $\Sigma' = \mathcal{T}_{\pi}^1 - \mathcal{T}_{\pi}^0 = \text{Pres}_{\pi, C'}^{-p} \cap \mathcal{T}_{\pi}^1$, and the restricted morphism $\bar{\kappa}^1 : \Sigma' \rightarrow \Sigma$ coincides with the morphism $\text{Pres}_{\pi, C'}^{-p} \cap \mathcal{T}_{\pi}^1 \rightarrow \bar{J}_{C'}^{-p} \cap \mathcal{S}_{\pi}^1$.

We must now distinguish two cases. If Q' lies in the regular locus of C' , that is, if π

forms a node or a cusp, then $\mathcal{S}_\pi^1 = \bar{J}_C^{-p}$, so the lower sequence is exact to the middle one in diagram (4.7). Furthermore, $\bar{\kappa}^1 = \bar{\kappa}$ is thus equal to $\text{Pres}_{\pi, C'}^{-p} \rightarrow \bar{J}_{C'}^{-p}$. According to diagram (2.8), the latter morphism is isomorphic to the projection $q_1 : \bar{J}_{C'}^{-p} \times Q' \rightarrow \bar{J}_{C'}^{-p}$. Using the Künneth formula, it is easy to see that $G_{q_1} = G_{\bar{\pi}} = G$. Hence $\underline{K}_\kappa = \underline{K}_\kappa^1 = G$, and we are done.

On the other hand, if Q' is supported at a double point P' of C' , then according to Proposition 3.15, we have an isomorphism $\text{Pres}_{\pi, C'}^{-p} \cap \mathcal{T}_\pi^1 \xrightarrow{\sim} J_{C'}^{-p} \times Q'$ and the morphism $\bar{\kappa}^1 : \Sigma' \rightarrow \Sigma$ is isomorphic to the projection $q_1 : J_{C'}^{-p} \times Q' \rightarrow J_{C'}^{-p}$. Here we need to be slightly more careful in showing that $\underline{K}_\kappa = G$, because $J_{C'}^{-p}$ is not proper. Note first that the morphism $\underline{K}_\kappa \rightarrow \underline{K}_\kappa^1$ is injective on T -points by Lemma 1.17. We already know that $G \rightarrow \underline{K}_\kappa$ is injective. Hence it suffices to show that \underline{K}_κ maps into the image of G in \underline{K}_κ^1 .

To achieve this, let Z be the closure in Pres_π^{-p} of $\text{Pres}_{\pi, C'}^{-p} \cap \mathcal{T}_\pi^1$. Recall from Proposition 3.15 that $\text{Pres}_{\pi, C'}^{-p}$ has two irreducible components E and E' , and that $E' \cap \mathcal{T}_\pi^1$ is empty. Hence Z is equal to the closure in Pres_π^{-p} of $E \cap \mathcal{T}_\pi^1$. We have a map $\mu : Z \rightarrow \bar{J}_{C'}^{-p}$ obtained by restricting κ . By Proposition 3.15 again, $Z \cap \mathcal{T}_\pi^1$ is isomorphic to $J_{C'}^{-p} \times Q'$ and the following diagram is Cartesian:

$$(4.8) \quad \begin{array}{ccc} J_{C'}^{-p} \times Q' & \longrightarrow & Z \\ q_1 \downarrow & & \downarrow \mu \\ J_{C'}^{-p} & \longrightarrow & \bar{J}_{C'}^{-p}. \end{array}$$

In particular, since $Z \cap \mathcal{T}_\pi^1$ has only one associated point, so does Z . The map $\sigma : \text{Pres}_{\pi, C'}^{-p} \rightarrow Q'$ defined in (2.7) restricts to a map $\sigma : Z \rightarrow Q'$, which is surjective since the induced map $Z \cap \mathcal{T}_\pi^1 = J_{C'}^{-p} \times Q' \rightarrow Q'$ is the projection. Hence Z is flat over Q' . The closed fiber of Z over Q' is equal to the reduced subscheme Z_{red} of Z , which is by construction equal to E_{red} . The latter is canonically isomorphic to $\text{Pres}_{\pi''}^{-p}$, where $\pi'' : C'' \rightarrow C'$ is the blow-up of C' at P' . In particular, since Z_{red} is proper and integral, we have $H^0(Z, \mathcal{O}_Z) = H^0(Q', \mathcal{O}_{Q'}) = k[\epsilon]/(\epsilon^2)$. Let \underline{EG}_μ be the functor of effective descent data of μ . From diagram (4.8) above, we deduce a morphism $\underline{EG}_\mu \rightarrow \underline{G}_{q_1} = \underline{K}_\kappa^1$. We claim that the image of this morphism contains the image of \underline{K}_κ in \underline{K}_κ^1 . Indeed, let \mathcal{L} be an invertible sheaf on $\bar{J}_{C'}^{-p} \times T$ such that $\kappa^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{\text{Pres}_\pi^{-p} \times T}$. Then the image of \mathcal{L} in $\underline{K}_\kappa^1(T)$ is $\mathcal{L}|_{\mathcal{S}_\pi^1}$, which corresponds to a descent datum ϕ^1 in $\underline{G}_{q_1}(T)$ under the identification $\underline{K}_\kappa^1 = \underline{G}_{q_1}$. On the other hand, we must also have $\mu^*(\mathcal{L}|_{\bar{J}_{C'}^{-p} \times T}) \xrightarrow{\sim} \mathcal{O}_{Z \times T}$, which gives an effective descent datum ϕ in $\underline{EG}_\mu(T)$. Clearly the restriction of ϕ in $\underline{K}_\kappa^1(T)$ is equal to ϕ^1 , and our claim is proved.

It thus remains to prove that the image of \underline{EG}_μ in \underline{K}_κ^1 is contained in G . Let $\phi \in \underline{EG}_\mu(T)$ correspond to a line bundle \mathcal{N} on $\bar{J}_{C'}^{-p} \times T$ and an isomorphism $\theta : \mu^* \mathcal{N} \xrightarrow{\sim} \mathcal{O}_{Z \times T}$. Then the image ϕ^1 of ϕ in \underline{K}_κ^1 corresponds to the line bundle $\mathcal{N}|_{\bar{J}_{C'}^{-p} \times T}$ and the restriction θ^1

of θ to $J_{C'}^{-p} \times Q' \times T$. However, $\text{Ker } q_1^* : \underline{\text{Pic}}_{J_{C'}^{-p}} \rightarrow \underline{\text{Pic}}_{J_{C'}^{-p} \times Q'}$ is trivial since q_1 has a section. Hence $\mathcal{N}|_{J_{C'}^{-p} \times T} \simeq \mathcal{O}_{J_{C'}^{-p} \times T}$ and $\phi^1 = (p_1^* \theta^1) \circ (p_2^* \theta^1)^{-1}$, where θ^1 corresponds to a unit in $H^0(J_{C'}^{-p} \times Q' \times T, \mathcal{O}_{J_{C'}^{-p} \times Q' \times T})$. But θ^1 is the restriction of θ , which corresponds (non-canonically) to a unit in $H^0(Z \times T, \mathcal{O}_{Z \times T})$ and the latter, as we've already seen, is isomorphic to $H^0(Q' \times T, \mathcal{O}_{Q' \times T})$. Hence ϕ^1 lies in $G(T)$, which finally proves the theorem.

□

Bibliography

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